Lectures in FEA

(Fundamentals)

Somenath Mukherjee

Scientist,
Structural Technologies Division,
National Aerospace Laboratories (NAL),
Bangalore,
Karnataka, India

Gangan Prathap

Director,
National Institute of Science and
Information Resources (NISCAIR),
New Delhi,
India

What everybody knows…

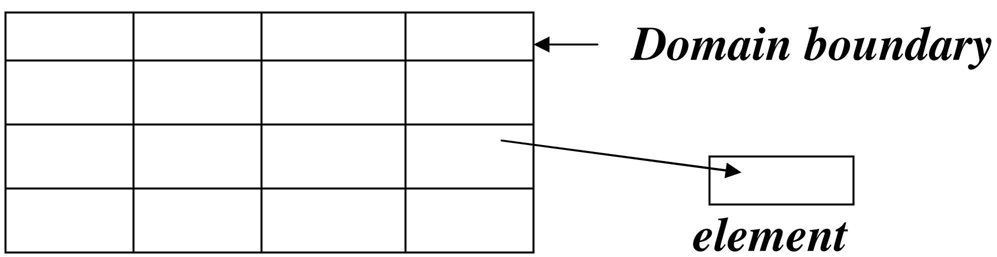
- **The Finite Element Method is a Numerical tool to solve differential equations (approximately) through discretization.**
- **Steps of FEM**

  (1) ***Discretisation of the continuum into elements.***

  (2) ***Element formulation using approximate solution for the differential equation.***

  (3) ***Element assembly.***

  (4) ***Solution of the system of equations, incorporating boundary conditions.***



- **Converge to more accurate results by using more elements (*i.e.* through finer discretization).**

Element formulation

Using approximate functions and variational principles for equilibrium, we establish $\{F^e_{nodal}\} = \lfloor K^e \rfloor \{\delta^e\}$

$$\{F^e_{nodal}\} = \{F^e_{applied}\} + \{R^e_{Connectivity}\}$$

$$F_{i\_node} = \sum_{j=1}^{N} k_{ij}.\delta_j \qquad N = Total \quad element\ DOF$$

k_{ij}=Force required at d.o.f. *i* for unit displacement for d.o.f. *j*,
with all other d.o.f. locked.

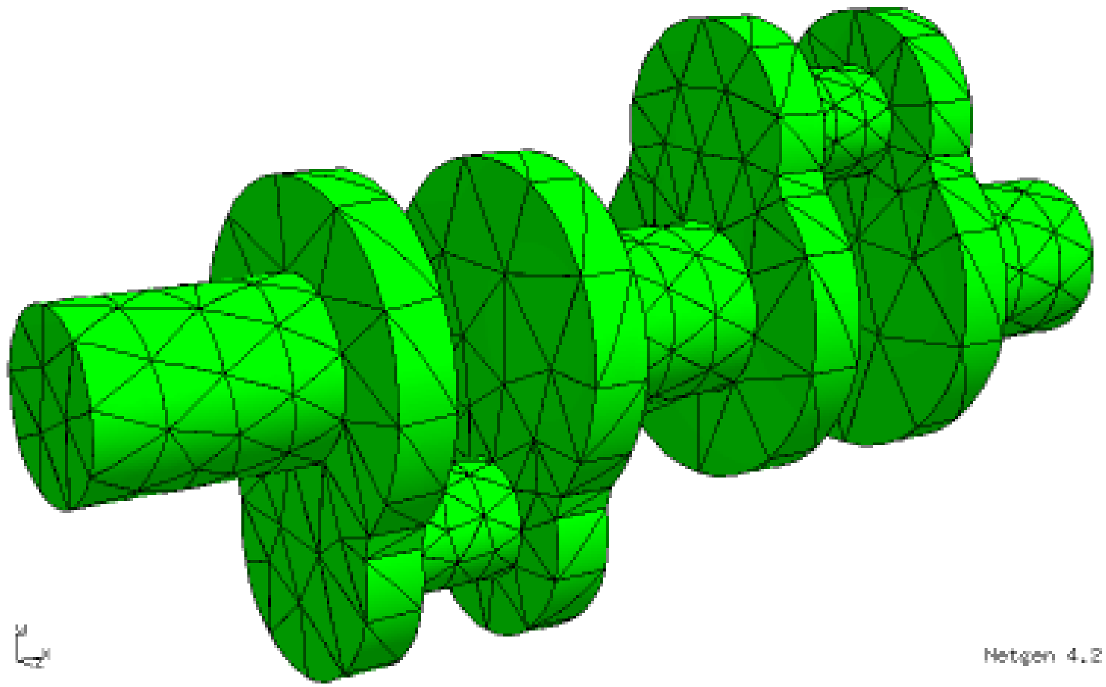
Assembly of elements: $Global \quad Force: \quad \{F^G\} = \sum_{e=1}^{N} \{F^e\}$

$$Global \quad Stiffness: \quad [K^G] = \sum_{e=1}^{N} [K^e]$$

Solve for displacements: $\{F^G\} = \lfloor K^G \rfloor \{\delta^G\}$

$$Boundary \quad conditions \quad satisfied$$

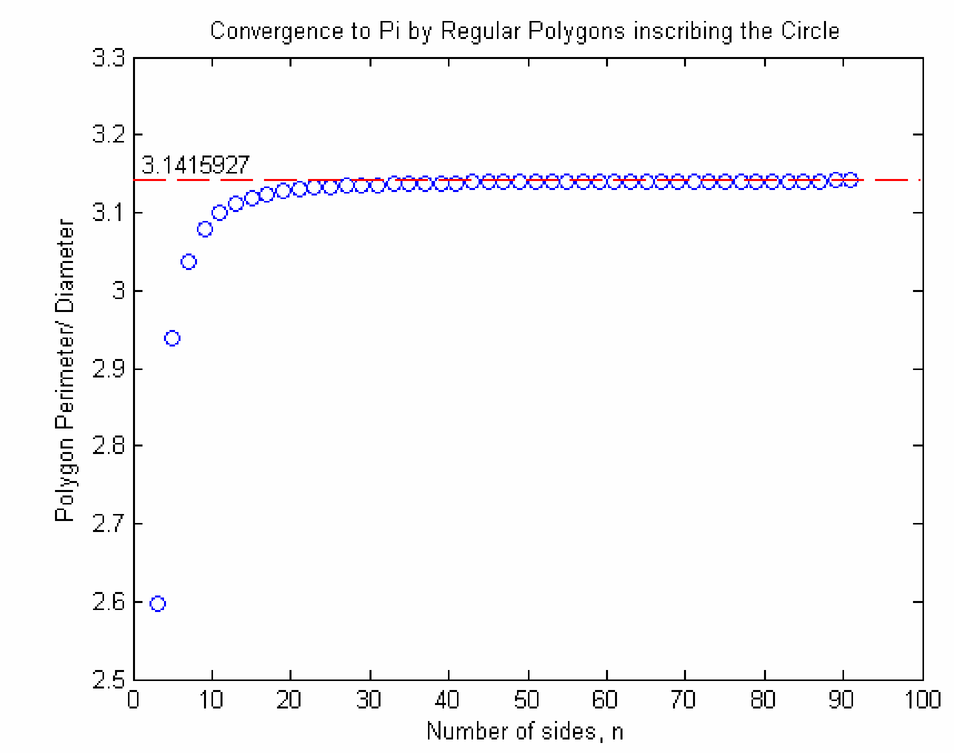
Estimate element Strains/Stresses from nodal displacements

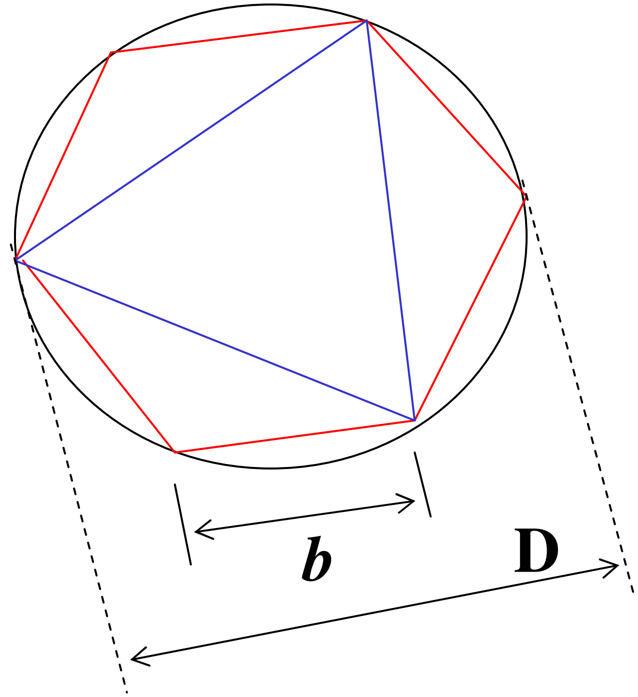


A machine system discretized into many finite elements

FEM - a tool for analysis through discretization

The Greeks had conceived the idea of realization through discretization(!) . How π was discovered.





$$Lim_{n\to\infty}\frac{Perimeter}{Diameter} = Lim_{n\to\infty}\frac{nb}{D} = 3.14159.... = \pi$$

$How?$

$$Lim_{n\to\infty}\frac{nb}{D} = Lim_{n\to\infty}[n.\sin(\pi/n)] = \pi$$

Lecture 1

Finite Element Analysis using simple Elements

Somenath Mukherjee

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Gangan Prathap

Director,
National Institute of Science and Information Resources (NISCAIR),
New Delhi,
India

Lecture 1

Finite Element Analysis using simple Elements

Chapters

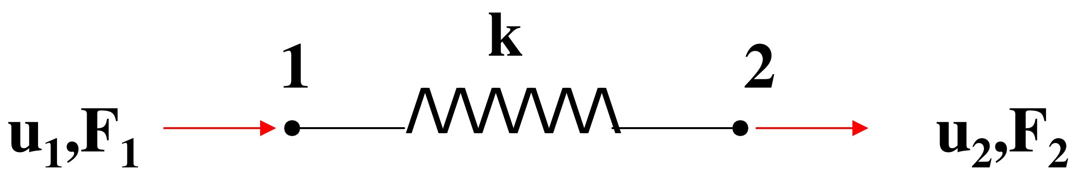
1. **Analysis of a system of assembly of springs.**
2. **Simple bar element.**
3. **Simple Classical Euler beam element.**
4. **Transformation of co-ordinates**

Lecture 1

Chapter 1

Analysis of a system of assembly of springs.

1.1 A single isolated spring element



Convention: Right hand direction is positive for forces/displacements.

$$Equilibrium:$$

$$F_1 = ku_1 - ku_2$$

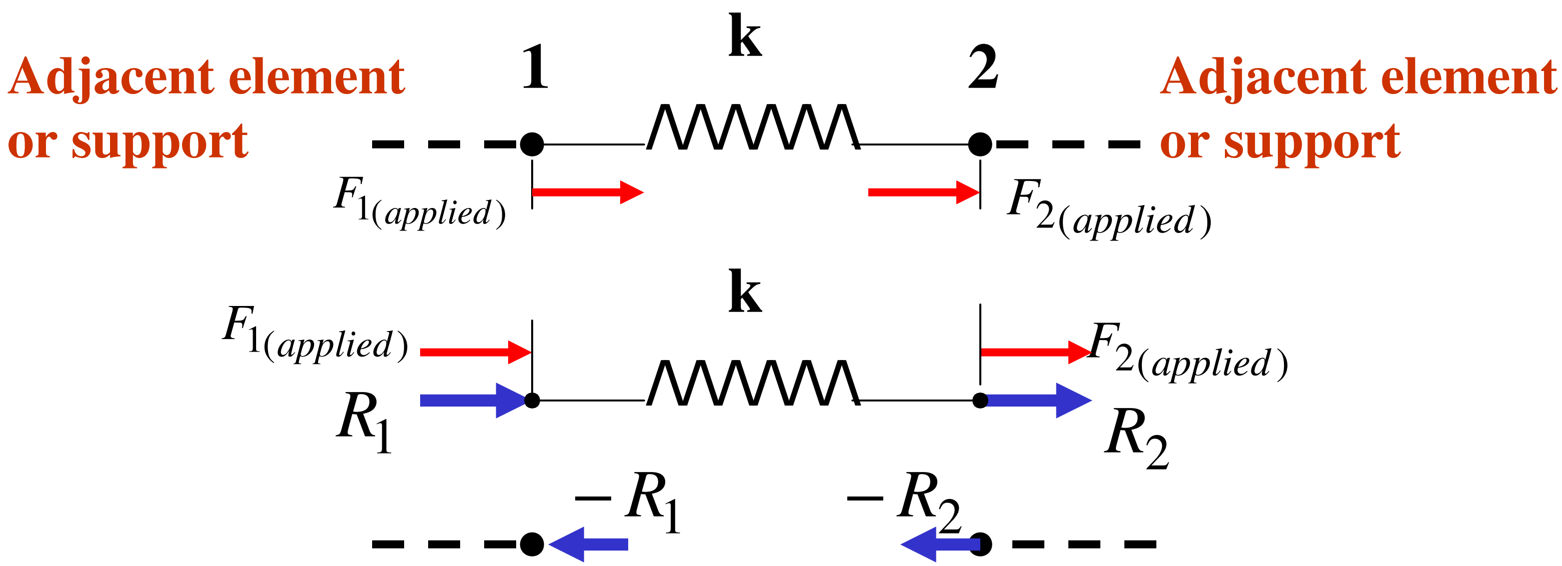
$$F_2 = -ku_1 + ku_2$$

$$i.e.$$

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad or \quad \{F^e\} = [K^e]\{u^e\} \qquad \textbf{(1.1)}$$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = [K^e] = spring \quad element \quad stiffness \quad matrix$$

1.2 A connected spring element



Convention: Right hand direction is positive for forces/displacements.

$$F_1 = F_{1(applied)} + R_1$$

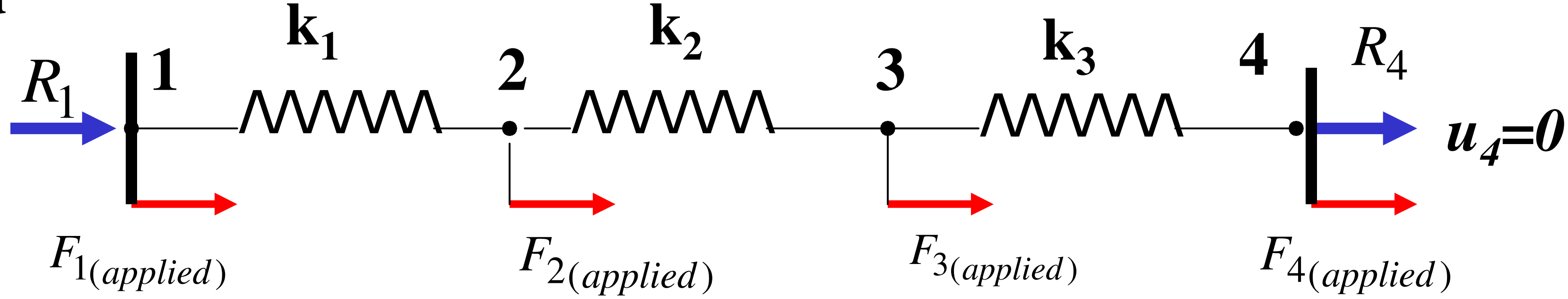
$$F_2 = F_{2(applied)} + R_2 \qquad R_1, R_2 \quad are \quad end \quad reactions$$

i.e.

$$\begin{Bmatrix} F_{1(applied)} + R_1 \\ F_{2(applied)} + R_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad or \quad \{F^e\} + \{R^e\} = [K^e]\{u^e\} \qquad \textbf{(1.2)}$$

1.3 An assembly of spring elements

Fixed end; ***u_1=0***



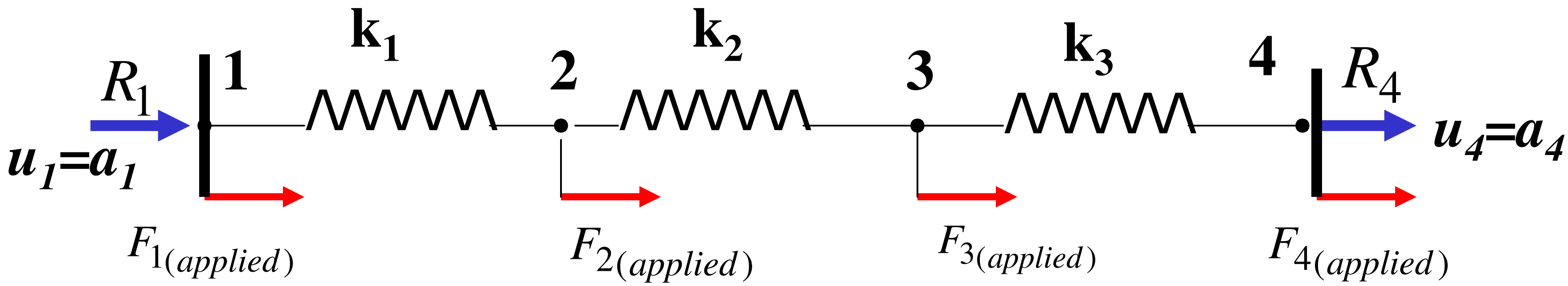
Assembly by adding corresponding parameters of same degree of freedom

Assembly :

$$\begin{Bmatrix} F_{1(applied)} + R_1 = ? \\ F_{2(applied)} \\ F_{3(applied)} \\ F_{4(applied)} + R_4 = ? \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 \\ 0 & 0 & k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = 0 \end{Bmatrix}$$

$$\boxed{\{F^G\} + \{R^G\} = [K^G]\{u^G\}} \qquad \textbf{(1.3)}$$

Note: For every degree of freedom, you know either the displacement, or the force. For every kinematic (displacement) boundary condition, there is an associated unknown force (reaction).



How do we incorporate <u>arbitrary boundary conditions</u> ?

$$u_1=a_1, \quad u_4=a_4$$

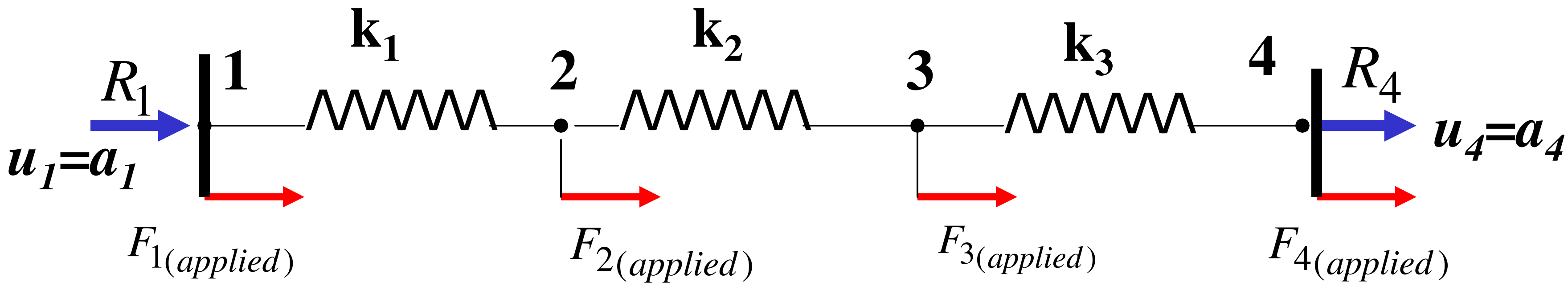
$$\begin{Bmatrix} F_{1(applied)}+R_1=? \\ F_{2(applied)} \\ F_{3(applied)} \\ F_{4(applied)}+R_4=? \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1=a_1 \\ u_2=? \\ u_3=? \\ u_4=a_4 \end{Bmatrix}$$

Solve

$$\begin{Bmatrix} F_{2(applied)}-(-k_1)u_1 \\ F_{3(applied)}-(-k_3)u_4 \end{Bmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

or

$$\{F_{new}\}=[K_{new}]\{u_{unknown}\} \qquad u_1=a_1, \quad u_4=a_4 \tag{1.4}$$



Solving for displacements

$u_1 = a_{1,} \qquad u_2 = a_4$

$Method \quad 1:$

$$\begin{Bmatrix} F_{2(applied)} - (-k_1)u_1 \\ F_{3(applied)} - (-k_3)u_4 \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

$Method \quad 2:$

$$\begin{Bmatrix} a_1 \\ F_{2(applied)} \\ F_{3(applied)} \\ a_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Boundary Reactions

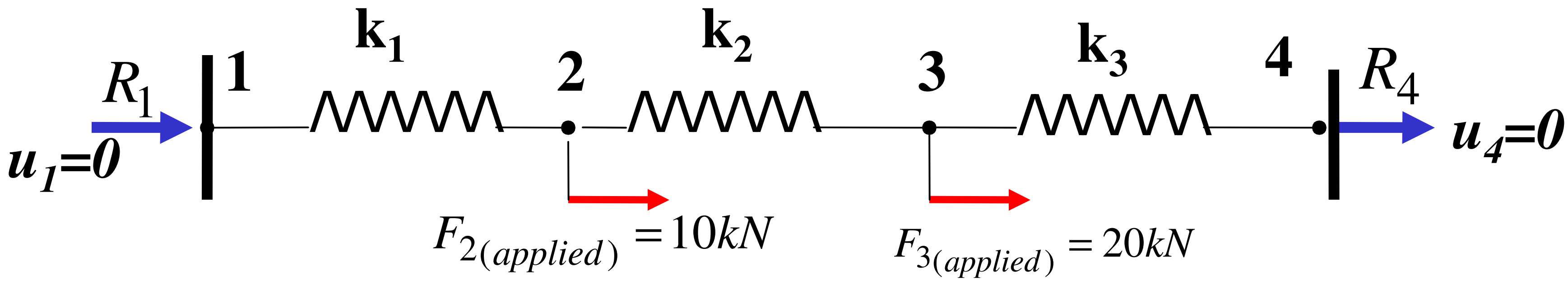
$$\{R^G\} = [K^G]\{u^G\} - \{F^G\}$$

$$\begin{Bmatrix} R_1 \\ 0 \\ 0 \\ R_4 \end{Bmatrix} = [K^G] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} - \begin{Bmatrix} F_{1(applied)} \\ F_{2(applied)} \\ F_{3(applied)} \\ F_{4(applied)} \end{Bmatrix}$$

(1.5)

$$R_1 = k_1 u_1 - k_1 u_2 - F_{1(applied)}$$

$$R_4 = -k_3 u_3 + k_3 u_4 - F_{4(applied)}$$



Example 1:

Solve for joint displacements and fixed support reactions for the spring assembly under the given loading. Given data:

k_1=1200 kN/m, k_2=1800 kN/m, k_3=1500 kN/m

F_2=10 kN, F_3= 20 kN

Use equation (1.4) and solve:

$$\begin{Bmatrix} 10 \\ 20 \end{Bmatrix} = \begin{bmatrix} 3000 & -1800 \\ -1800 & 3300 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Displacements: $\boxed{u_2 = 0.0104 \text{ m}, \quad u_3 = 0.0117 \text{ m}}$

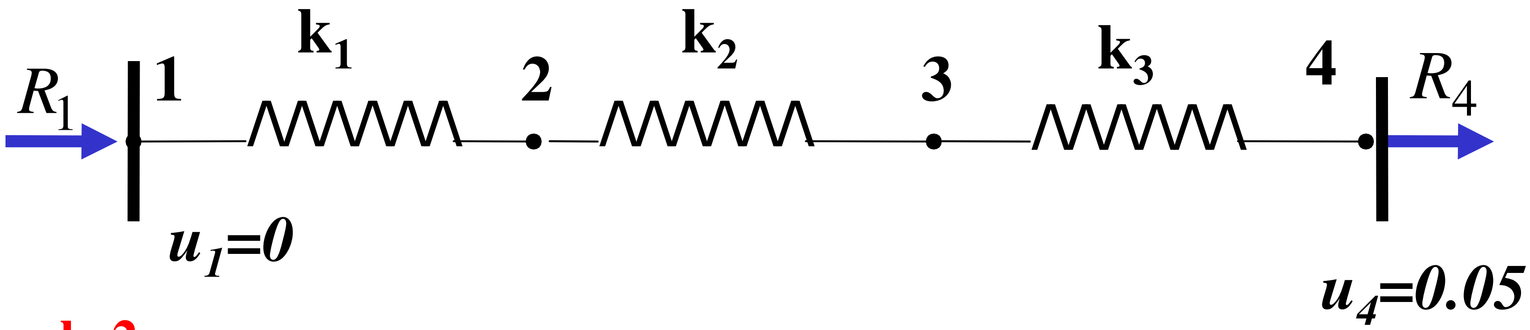
Use equation (1.5) :

$$R_1 = 0 - 1200u_2 - 0$$
$$R_4 = -1500u_3 + 0 - 0$$

Support Reactions: $\boxed{R_1 = -12.43 \text{ kN } (\longleftarrow), \; R_4 = -17.57 \text{ kN } (\longleftarrow)}$

Note: Applied forces and displacements are rightwards (positive). Support reactions are leftwards (negative); opposite to the direction shown in the figure.

Equilibrium: $F_2 + F_3 + R_1 + R_4 = 0$



Example 2:

Solve for joint displacements and support reactions for the spring assembly if the left end is fixed and the right end is pulled out by u_4=0.05 m. No other forces are applied.

k_1=1200 kN/m, k_2=1800 kN/m, k_3=1500 kN/m

Use equation (1.4) and solve:

$$\begin{Bmatrix} 0 \\ 1500.u_4 \end{Bmatrix} = \begin{bmatrix} 3000 & -1800 \\ -1800 & 3300 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Displacements:

$$\boxed{u_2 = 0.0203 \text{ m}, \quad u_3 = 0.0338 \text{ m}}$$

Use equation (1.5) :

$$R_1 = -1200 u_2$$
$$R_4 = -1500 u_3 + 1500 u_4$$

Support Reactions:

$$\boxed{R_1 = -24.32 \text{ kN } (\leftarrow), \quad R_4 = 24.32 \text{ kN } (\rightarrow)}$$

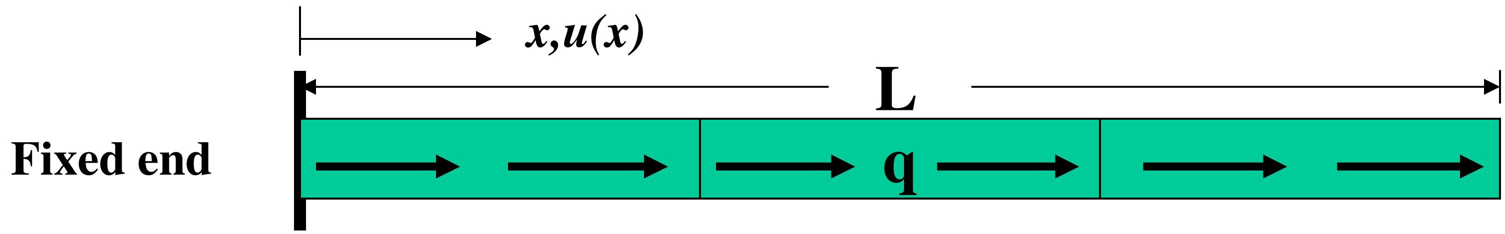
Note: One should apply a force of 24.32 kN rightward at the right end to achieve an end displacement of u_4=0.05 m rightward.

Lecture 1

Chapter 2

Simple bar element

2.1 Classical formulation of a bar under axial distributed loading intensity *q(x)*



Governing differential equation:

$$\boxed{-\frac{d}{dx}\left(EA\frac{du}{dx}\right)=q(x)} \qquad \textbf{(2.1)}$$

Strain: $\varepsilon = \dfrac{du}{dx}$

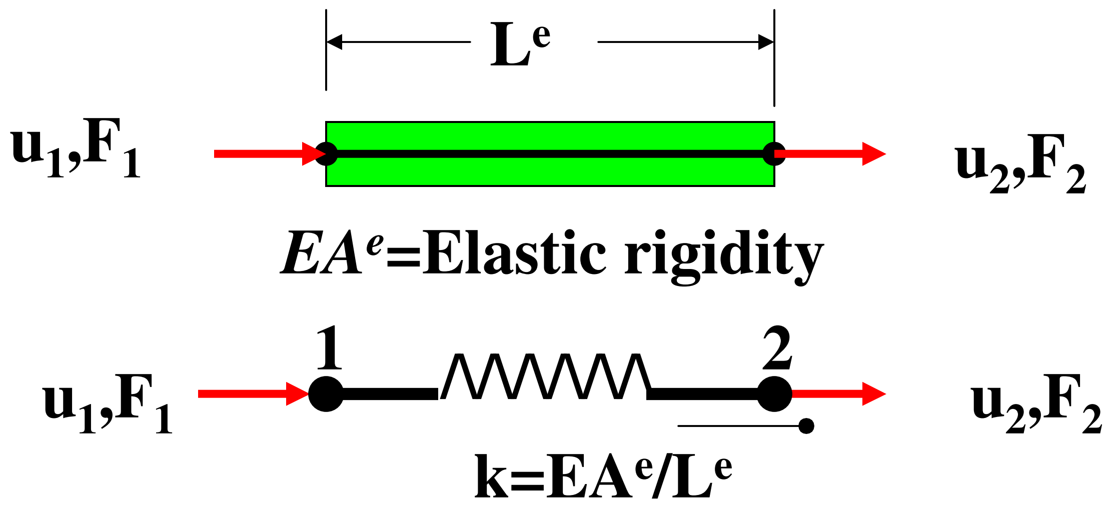
Stress: $\sigma = E\varepsilon = E\dfrac{du}{dx}$

Stress resultant : $P = \sigma A = EA\dfrac{du}{dx}$ **(2.2)**

Strain Energy:

$$U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx \qquad \textbf{(2.3)}$$

2.2 A simple bar element as an equivalent spring element



E=Elastic Modulus **A^e=Constant section area**

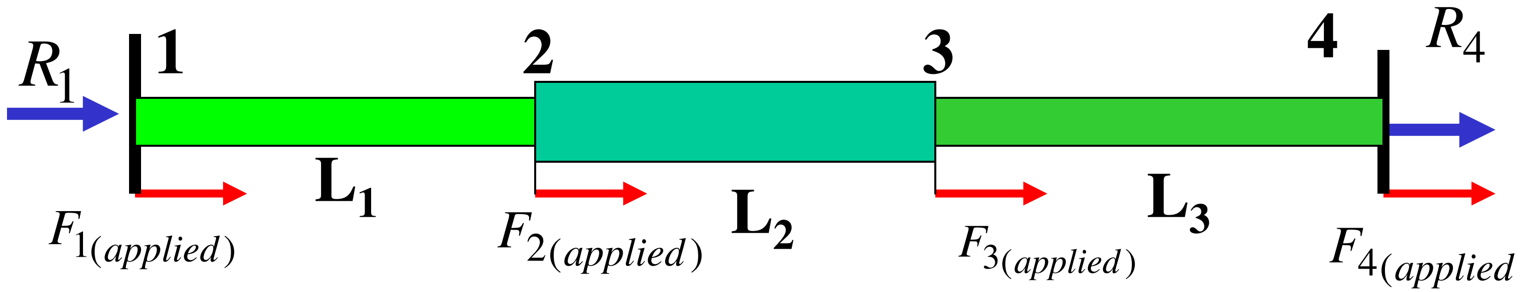
Equilibrium :

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} EA^e/L^e & -EA^e/L^e \\ -EA^e/L^e & EA^e/L^e \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [K^e] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \textbf{(2.4)}$$

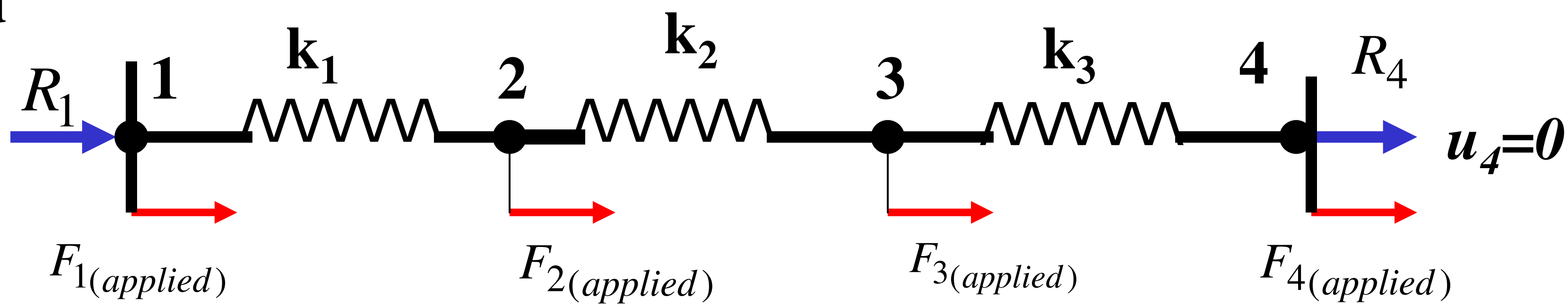
$[K^e] = bar \quad element \quad stiffness \quad matrix$

For connected element: $$\boxed{\{F^e\} + \{R^e\} = [K^e]\{u^e\}} \qquad \textbf{(2.5)}$$

2.3 An assembly of bar elements as spring elements



Fixed end;
u_1=0



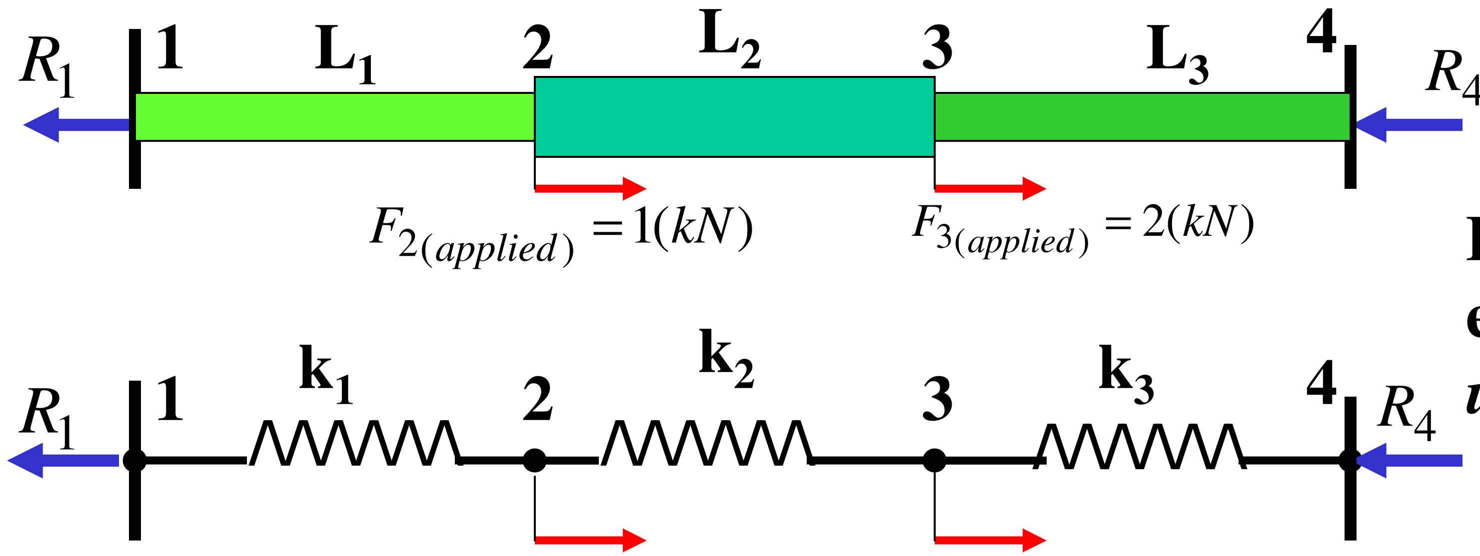
Equivalent spring stiffnesses $k_1=EA_1/L_1$ $k_2=EA_2/L_2$ $k_3=EA_3/L_3$

Assembly :

$$\begin{Bmatrix} F_{1(applied)}+R_1=? \\ F_{2(applied)} \\ F_{3(applied)} \\ F_{4(applied)}+R_4=? \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & k_3 & k_3 \end{bmatrix} \begin{Bmatrix} u_1=0 \\ u_2=? \\ u_3=? \\ u_4=0 \end{Bmatrix}$$

$$\boxed{\{F^G\}+\{R^G\}=[K^G]\{u^G\}} \qquad \textbf{(2.6)}$$

Fixed end; $u_1=0$



Fixed end; $u_4=0$

Example 3:

Solve for joint displacements and fixed support reactions for the bar assembly under the given loading. F_2**=1 kN,** F_3**= 2 kN.**

Young's Modulus E**=200 GPa,**

$A_1=6(10^{-6})\text{m}^2$, $A_2=9(10^{-6})\text{m}^2$, $A_3=7.5(10^{-6})\text{m}^2$, $L_1=L_2=L_3=0.1\text{m}$.

Equivalent spring stiffnesses

$k_1=EA_1/L_1=$ **12000 kN/m ,** $k_2=EA_2/L_2=$**18000 kN/m** $k_3=EA_3/L_3=$**15000 kN/m**

Solution as in Example 1 (for springs)

u_2**=0.000104 m,** u_3**= 0.000117 m**	R_1**= -1.243 kN (←) ,** R_4**= -1.757 kN (←)**

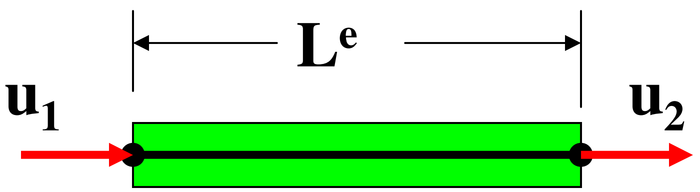
2.4 Strain and stress resultant in bar element

Element Strain

$$\varepsilon^h = \frac{u_2 - u_1}{L^e} = \begin{bmatrix} \frac{-1}{L^e} & \frac{1}{L^e} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\varepsilon^h = [B]\{\delta^e\} \qquad \textbf{(2.7)}$$

$$[B] = Strain - displacement \quad matrix$$



EA^e=**Elastic rigidity**

Element Stress Resultant

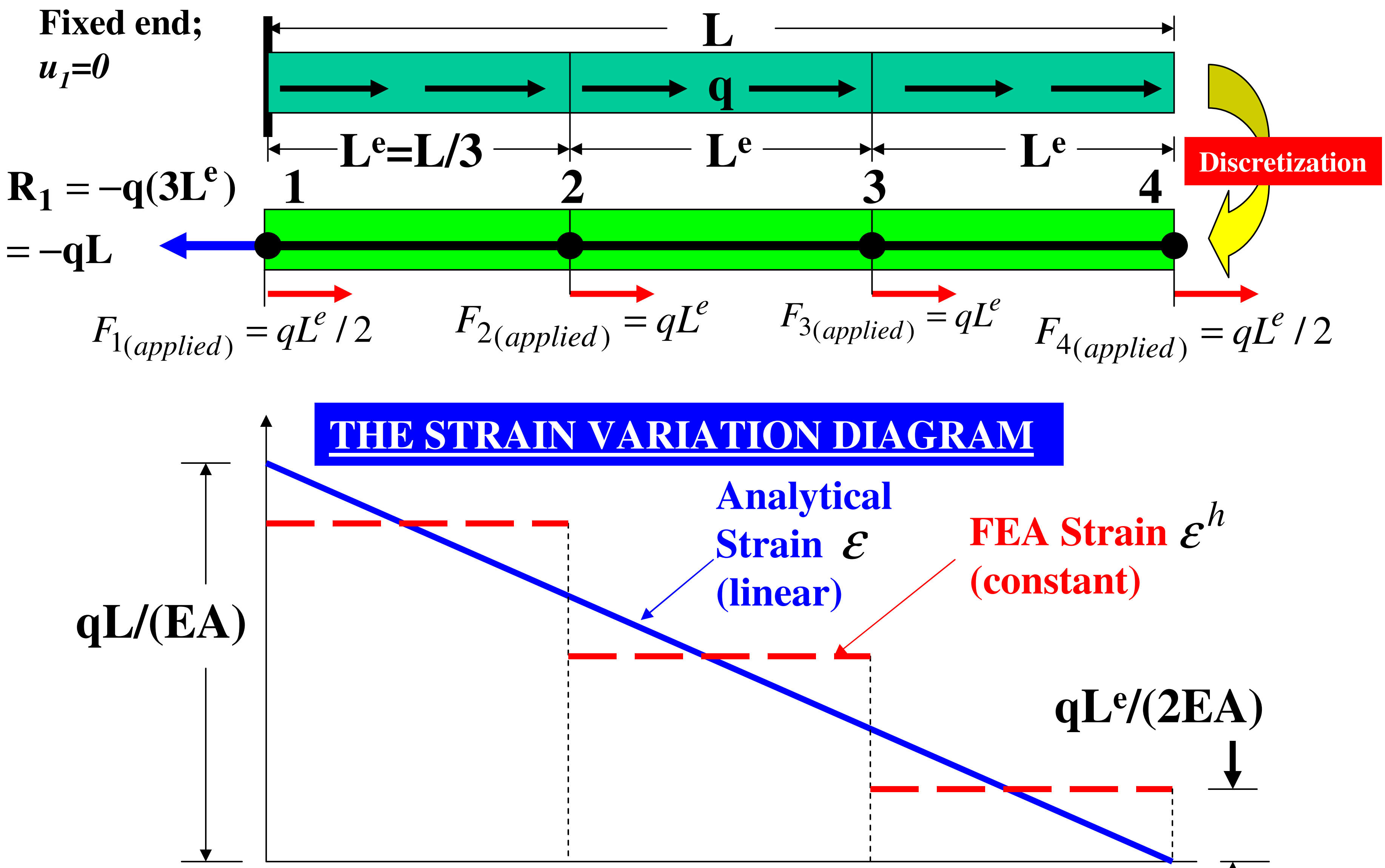
$$\{P^e\} = EA^e.[B]\{\delta^e\}$$

$$In \quad general$$

$$\{\sigma^h\} = [D][B]\{\delta^e\} \qquad \textbf{(2.8)}$$

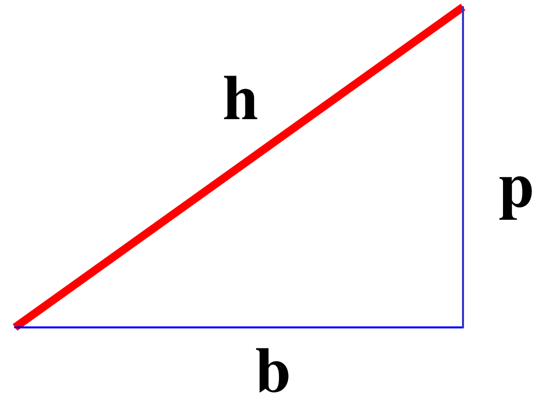
$$[D] = Element \quad elastic \quad rigidity \quad matrix$$

2.5 Analysis of a bar under u.d.l. *q* using FEM



FEA gives the best-fit to the analytical strain. Why ?

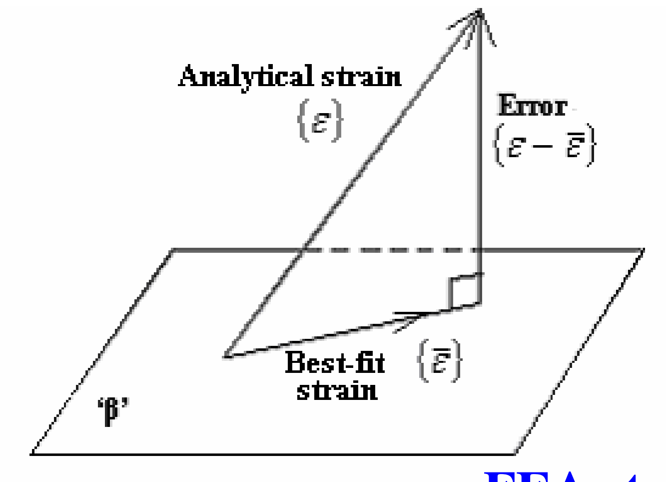
Pythagoras



$$h^2=b^2+p^2$$

2.6 FEA as best fit ?

The best fit strain vector is the orthogonal projection of the analytical strain vector, onto a given subspace.



$$\bar{\varepsilon} = Best - fit$$

A best fit satisfies the following Theorem (Pythagoras)

$$\left\|\varepsilon - \bar{\varepsilon}\right\|^2 = \left\|\varepsilon\right\|^2 - \left\|\bar{\varepsilon}\right\|^2$$

***apostiori* analysis of FEA results shows that**
FEA strain matches the best-fit strain at the element level.

$$\boxed{\varepsilon^h = \bar{\varepsilon}}$$

Thus

$$\boxed{\left\|\varepsilon - \varepsilon^h\right\|^2 = \left\|\varepsilon\right\|^2 - \left\|\varepsilon^h\right\|^2} \quad \textbf{(2.9)}$$

i.e. The Energy of the Error= Error of the Energies

We need a paradigm to show -
How FEA turns out as the best fit to the analytical solution.

The Bilinear Symmetric Form and the inner product

- Element Strain Vector by FEA: $\{\varepsilon^h\} = [B]\{\delta^e\}$
- Analytical element strain vector : $\{\varepsilon\}$
- Bilinear form and Inner product definition :

$$\boxed{a(u^h, u)^e = \int_e \{\varepsilon^h\}^T [D]\{\varepsilon\} dx = < \varepsilon^h, \varepsilon >}$$

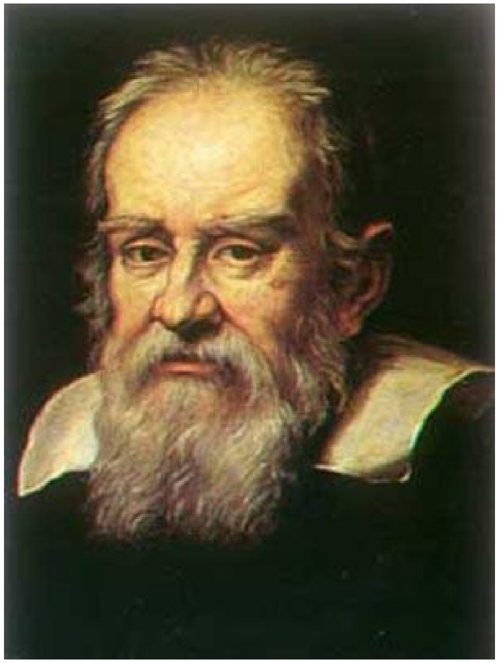
$$\boxed{a(u^h, u^h)^e = \int_e \{\varepsilon^h\}^T [D]\{\varepsilon^h\} dx = < \varepsilon^h, \varepsilon^h > = \left\|\varepsilon^h\right\|^2}$$

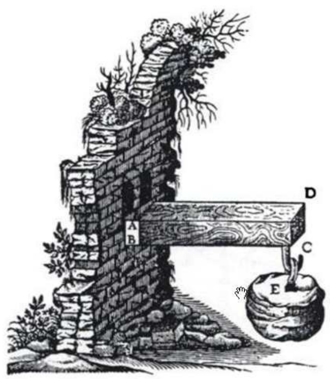
Here [D] is the element rigidity matrix

Lecture 1

Chapter 3

Simple Classical Euler beam element

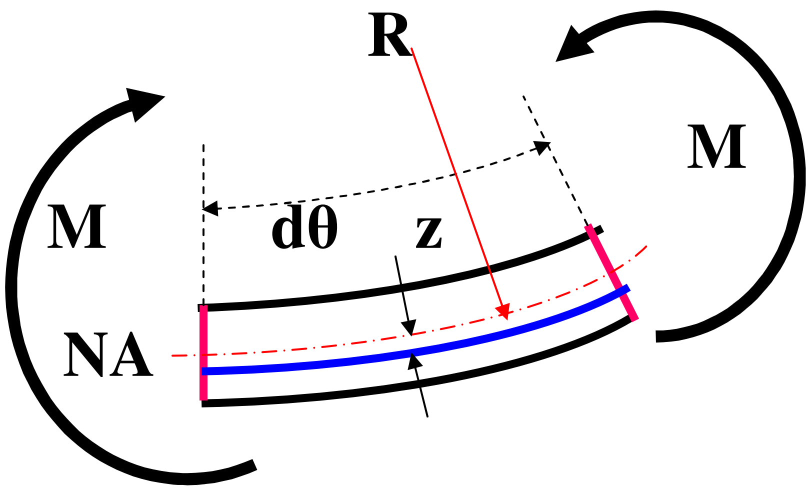


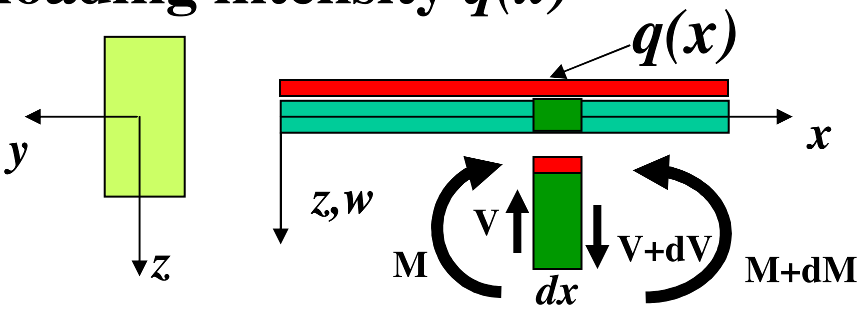


Galileo was the first person to perform experiments on a cantilever beam

Euler derived the equation for the thin beam

3.1 Classical formulation of a the Euler beam under transverse distributed loading intensity *q(x)*





Equilibrium:

$$V = \frac{dM}{dx}, \qquad q = -\frac{dV}{dx} \qquad \Rightarrow \qquad -\frac{d^2M}{dx^2} = q$$

Strain: $\varepsilon = \dfrac{du}{dx} = \dfrac{(R+z)d\theta - Rd\theta}{R.d\theta} = \dfrac{z}{R}$

Stress: $\sigma = E\varepsilon = E\dfrac{z}{R}$

Governing differential equation:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = q \qquad (3.1)$$

NA passes through section centroid:

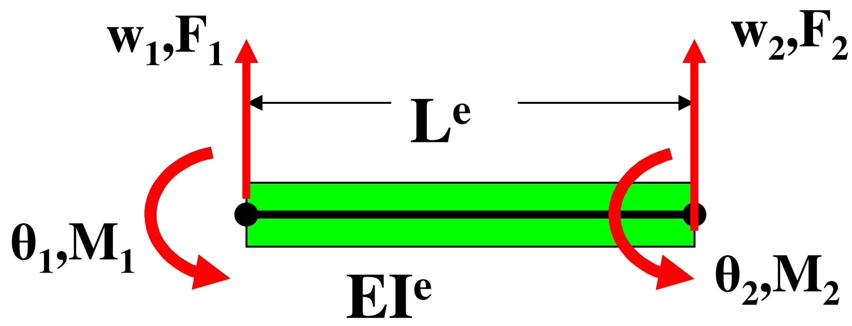
$$P = \int_A \sigma.dA = \frac{E}{R}\int_A z.dA = 0 \qquad \Rightarrow \qquad \bar{z} = 0$$

Bending moment and bending stress:

$$M = \int_A \sigma.z.dA = \frac{E}{R}\int_A z^2.dA = \frac{EI}{R} \qquad \sigma = E\frac{z}{R} = \frac{M}{I}z \qquad I = \int_A z^2.dA$$

3.2 Basic formulation, properties and use of the Euler beam element (using direct method)

EI^e=Elastic bending rigidity



$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \rightarrow \boxed{\{F^e{}_{applied}\} + \{R^e\} = [K^e]\{\delta^e\}} \quad \textbf{(3.2)}$$

$$\left[K^e\right] = \begin{bmatrix} 12(EI^e/L^{e^3}) & 6(EI^e/L^{e^2}) & -12(EI^e/L^{e^3}) & 6(EI^e/L^{e^2}) \\ 6(EI^e/L^{e^2}) & 4(EI^e/L^e) & -6(EI^e/L^{e^2}) & 2(EI^e/L^e) \\ -12(EI^e/L^{e^3}) & -6(EI^e/L^{e^2}) & 12(EI^e/L^{e^3}) & -6(EI^e/L^{e^2}) \\ 6(EI^e/L^{e^2}) & 2(EI^e/L^e) & -6(EI^e/L^{e^2}) & 4(EI^e/L^e) \end{bmatrix} \quad \textbf{(3.3)}$$

- **The stiffness matrix is symmetric.** $k_{ij}=k_{ji}$
- **C1= - C3 and R1= - R3**
- **C1=(C2+C4)/L^e**
- **R1=(R2+R4)/L^e**
- **Det [K^e]=0**

WHY ?

I. DERIVATION OF THE ELEMENT STIFFNESS MATRIX (BY DIRECT METHOD)

Assumptions:

— All assumptions for the classical beam theory apply.

viz. transverse shear strain is very small, and hence can be ignored.

— Forces act only at the nodes.

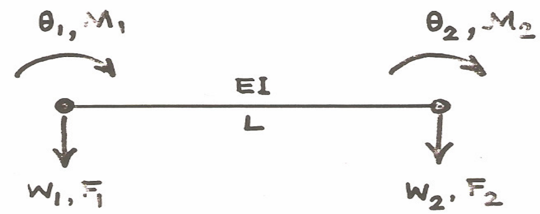


FIG 7

The element stiffness matrix $[K^e]$;

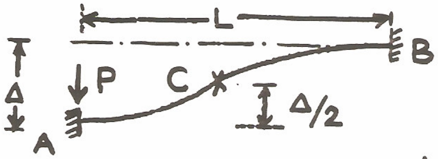
$$\{F^e\} = [K^e]\{\delta^e\}$$

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{Bmatrix}$$

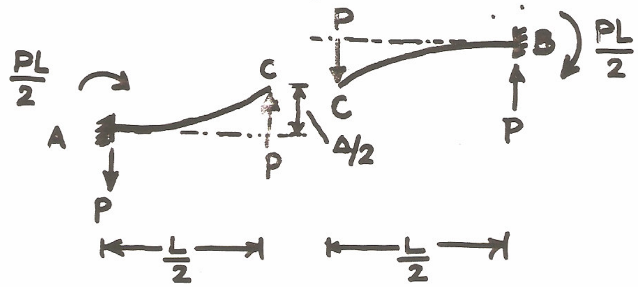
K_{ij} is the force reqd. at d.o.f. 'i' for a unit displacement at 'j' d.o.f., all d.o.f(s) other than 'j' being 'LOCKED'

TWO CASES OF DEFLECTIONS.

CASE A:



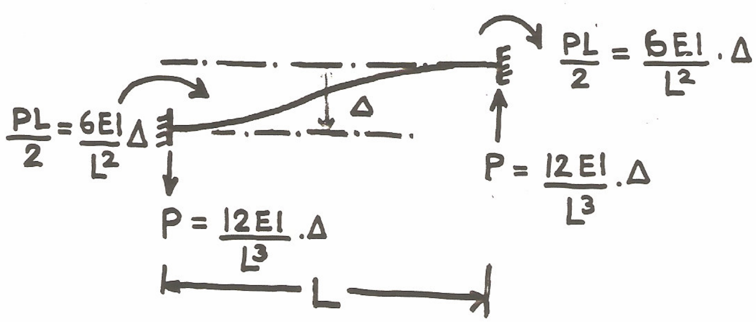
Since point C is a point of 'contraflexure' (No curvature), we can model AB as two symmetric cantilevers.



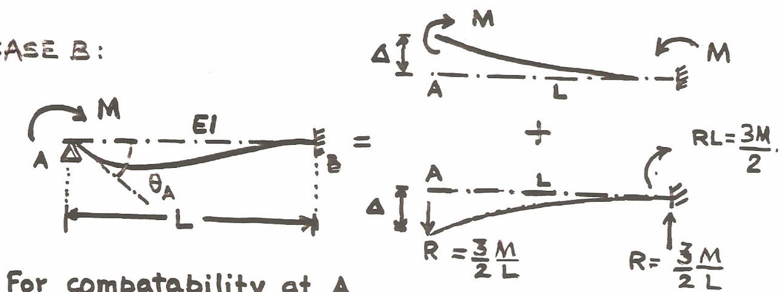
$$\therefore \frac{\Delta}{2} = \frac{P(L/2)^3}{3EI} \Rightarrow P = \frac{12EI}{L^3}.\Delta$$

Moment at Ends: $\frac{PL}{2} = \frac{6EI}{L^2}.\Delta$

Finally, by joining the segments:



CASE B:



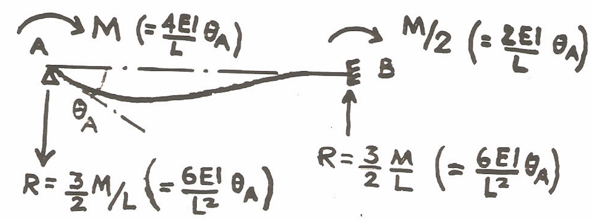
For compatability at A

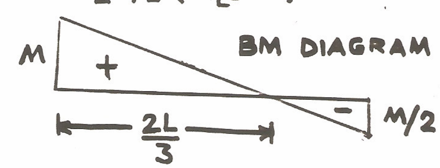
$$\Delta = \frac{ML^2}{2EI} = \frac{RL^3}{3EI}$$

$$\Rightarrow \quad R = \frac{3}{2}.\frac{M}{L}$$

$$RL = \frac{3}{2}M$$

By superposition:





By Moment area Method

θ_A = Area of $\frac{BM}{EI}$ Diag. between A and B.

$$\theta_A = M.\frac{2L}{3}.\frac{1}{2} - \frac{M}{2}.\frac{L}{3}.\frac{1}{2} = \frac{ML}{4EI}$$

$$M = \frac{4EI}{L}\theta_A \quad ; \qquad R = \frac{3}{2}\frac{M}{L} = \frac{6EI}{L^2}\theta_A$$

COLUMN 1 (C1) of the matrix $[K^e]$

Impose unit value on w_1 $(w_1 = 1)$

Lock θ_1, w_2, θ_2 $(\theta_1 = w_2 = \theta_2 = 0)$

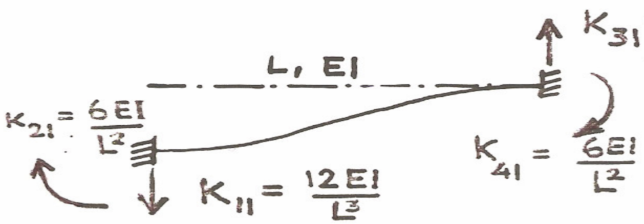


FIG A

$K_{11} = \frac{12EI}{L^3}$	**Equilibrium:**
$K_{21} = \frac{6EI}{L^2}$	$K_{11} = -K_{31}$
$K_{31} = \frac{-12EI}{L^3}$	$K_{21} + K_{41} = K_{11} \cdot L$
$K_{41} = \frac{6EI}{L^2}$	

COLUMN 2 (C2) of the matrix $[K^e]$

Impose unit value on θ_1 $(\theta_1 = 1)$

Lock w_1, w_2, θ_2 $(w_1 = w_2 = \theta_2 = 0)$

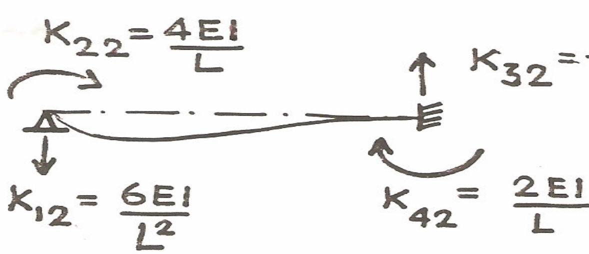


FIG B

$K_{12} = \frac{6EI}{L^2}$	**Equilibrium:**
$K_{22} = \frac{4EI}{L}$	$K_{12} = -K_{32}$
$K_{32} = \frac{-6EI}{L^2}$	$K_{22} + K_{42} = K_{12} L$
$K_{42} = \frac{2EI}{L}$	

COLUMN 3 (C3) of the matrix $[K^e]$

Impose unit value on w_2 $(w_2 = 1)$

Lock w_1, θ_1 and θ_2 $(w_1 = \theta_1 = \theta_2 = 0)$

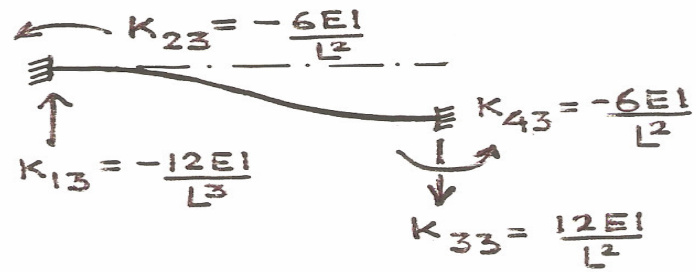


FIG C

$K_{13} = -\frac{12EI}{L^3}$	Equilibrium:
$K_{23} = -\frac{6EI}{L^2}$	$K_{13} = -K_{33}$
$K_{33} = \frac{12EI}{L^2}$	$K_{23} + K_{43} = K_{13}L$
$K_{43} = \frac{-6EI}{L^2}$	

COLUMN 4 (C4) of the matrix $[K^e]$

Impose unit value on θ_2 $(\theta_2 = 1)$

Lock w_1, θ_1 and w_2 $(w_1 = \theta_1 = w_2 = 0)$

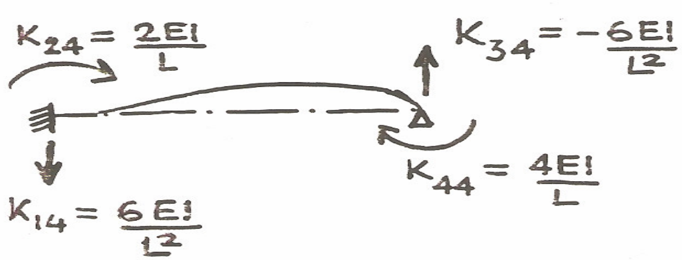


FIG D

$K_{14} = \frac{6EI}{L^2}$	Equilibrium:
$K_{24} = \frac{2EI}{L}$	$K_{14} = -K_{34}$
$K_{34} = -\frac{6EI}{L^2}$	$K_{24} + K_{44} = K_{14}L$
$K_{44} = \frac{4EI}{L}$	

II. PROPERTIES OF THE ELEMENT STIFFNESS MATRIX $[K^e]$

$$[K^e] = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

PROPERTY 1: THE ELEMENT STIFFNESS MATRIX IS SQUARE

This is so because we have as many nodal forces as the d.o.f.(s) for displacements.

PROPERTY 2. THE DIAGONAL ELEMENTS ARE ALL POSITIVE

If of the total d.o.f(s), only i-th d.o.f. is given unit displacement, all other d.o.f(s) being locked, then work done by the external nodal forces is contributed only by this i-th d.o.f.

This work done = Strain energy = positive
Hence the corresponding force has to in the same direction as the displacement of the i-th d.o.f.

This makes K_{ii} always positive.

PROPERTY 3: THE STIFFNESS MATRIX IS SYMMETRIC.

Suppose two different force vectors $\{F\}$ and $\{F^*\}$ cause corresponding displacements $\{\delta\}$ and $\{\delta^*\}$ (respectively) on the same system, of stiffness $[K]$.

By Maxwell's Reciprocal Theorem (valid for all linearly elastic conservative systems)

$$\{\delta\}^T\{F^*\} = \{\delta^*\}^T\{F\}$$

i.e.

$$\{\delta\}^T[K]\{\delta^*\} = \{\delta^*\}^T[K]\{\delta\}$$

For this to be valid, with $\{\delta^*\} \neq \{\delta\}$ in general,

$$\boxed{\begin{gathered}[K]^T = [K] \\ k_{ij} = k_{ji}\end{gathered}}$$

Conclusion: All linear and conservative systems have symmetric stiffness matrices.

This is also true at element level.

$$[K^e]^T = [K^e]$$

PROPERTY 4: COLUMN 1 = − COLUMN 3

ROW 1 = − ROW 3

From Figs. A – D for vertical equilibrium:

$$\boxed{K_{1j} = -K_{3j}} \quad \text{(ROW 1 = − ROW 3)}$$

By symmetry,

$$K_{3j} = K_{j3}$$

$$K_{1j} = K_{j1}$$

$$\therefore \boxed{K_{j1} = -K_{j3}} \quad \text{(COLUMN 1 = − COLUMN 3)}$$

PROPERTY 5: $\text{COLUMN } 1 = \frac{(\text{COLUMN } 2 + \text{COLUMN } 4)}{L}$

From Figs 10-13, for moment equilibrium:

$$K_{1j} L = K_{2j} + K_{4j}$$

By symmetry, $K_{1j} = K_{j1}$; $K_{2j} = K_{j2}$; $K_{4j} = K_{j4}$

$$\therefore \quad K_{j1} \cdot L = K_{j2} + K_{j4}$$

HENCE

$$\boxed{K_{1j} = \frac{K_{2j} + K_{4j}}{L}} \quad : R1 = \frac{R2 + R4}{L}$$

&

$$\boxed{K_{j1} = \frac{K_{j2} + K_{j4}}{L}} \quad : C1 = \frac{C2 + C4}{L}$$

PROPERTY 6: THE ELEMENT STIFFNESS MATRIX IS SINGULAR

i.e. $\boxed{\det [K^e] = 0}$

Any one of the properties 4 & 5 is a sufficient condition to make $[k^e]$ singular.

Since $[k^e]$ is singular, we have as consequences

(i) $\frac{1}{2} \{\delta\}^T [k^e] \{\delta\} = 0 \Rightarrow$ Strain energy is zero

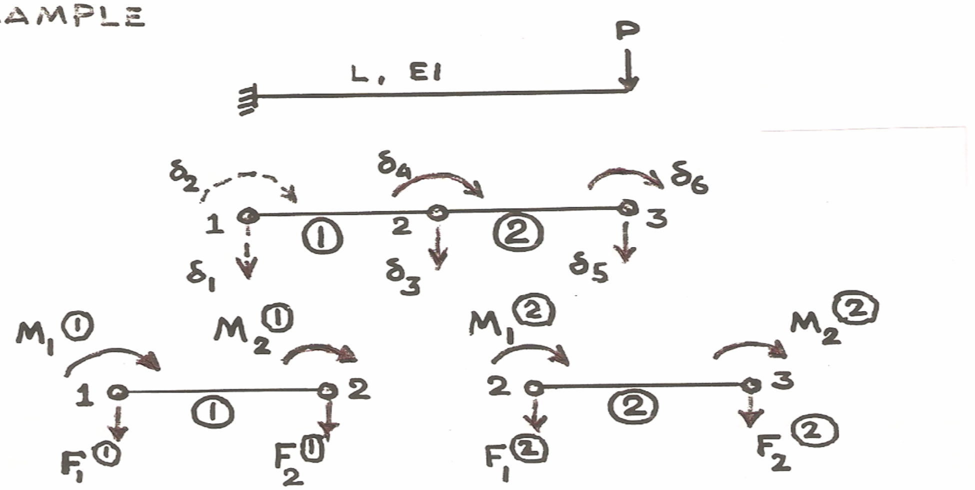
(ii) Eigenvalues of $[k^e]$ are zero.

Both (i) & (ii) are for rigid body displacements.

III. ASSEMBLY OF ELEMENTS

- Summation of stiffness coefficients of the element stiffness matrices for the nodal d.o.f(s) at interelement boundary nodes
- Summation of all forces at the common d.o.f(s) of interelement boundary nodes.

EXAMPLE



$$\begin{Bmatrix} F_1^{(1)} \\ M_1^{(1)} \\ F_2^{(1)} + F_1^{(2)} \\ M_2^{(1)} + M_1^{(2)} \\ F_2^{(2)} \\ M_2^{(2)} \end{Bmatrix} = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} & 0 & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & 0 & 0 \\ K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)}+K_{11}^{(2)} & K_{34}^{(1)}+K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\ K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)}+K_{21}^{(2)} & K_{44}^{(1)}+K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\ 0 & 0 & K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\ 0 & 0 & K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}$$

IV. BOUNDARY CONDITIONS / SUPPORT REACTIONS :

- From properties of determinants, global stiffness matrix $[K_G]$ is singular

$\Rightarrow$ Rigid body displacements.

- Boundary conditions for flexible modes need to be given.

IMPORTANT :

- For every nodal point, for every d.o.f.

WE EITHER KNOW THE NODAL FORCES (TOTAL)

OR

WE KNOW THE DISPLACEMENTS

(This is a fundamental idea in Variational Calculus.).

- Thus if we prescribe, as boundary conditions, pre-assigned values to chosen d.o.f.(s), it implies that we have no knowledge <u>a priori</u> of the corresponding total force for the same d.o.f.

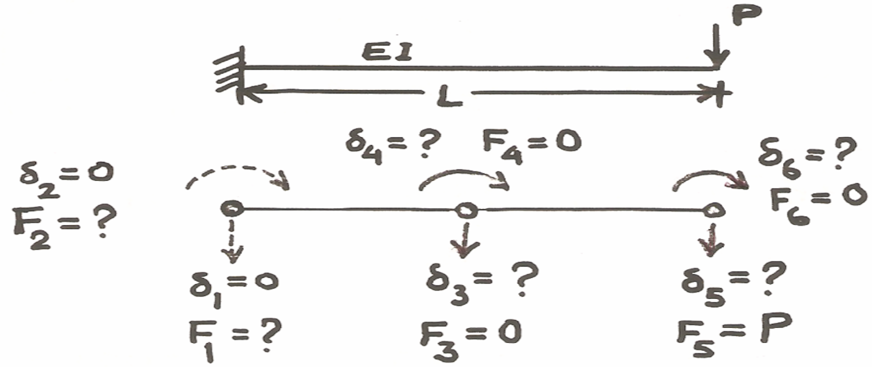
- Conversely, the unknown displacements are those that correspond to these d.o.f.(s) for which the nodal forces are prescribed <u>a priori</u>

METHODS TO INCORPORATE BOUNDARY CONDITIONS.

(A) DIRECT METHOD OF ELIMINATION

(B) PENALTY METHOD

We present the cantilever beam with tip load, with discretization into two elements,



A. THE DIRECT METHOD OF ELIMINATION

$$\begin{Bmatrix} F_1 = ? \\ F_2 = ? \\ F_3 = 0 \\ F_4 = 0 \\ F_5 = P \\ F_6 = 0 \end{Bmatrix}_G = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ & & K_{33} & K_{34} & K_{35} & K_{36} \\ & \text{symmetric} & & K_{44} & K_{45} & K_{46} \\ & & & & K_{55} & K_{56} \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \delta_1 = 0 \\ \delta_2 = 0 \\ \delta_3 = ? \\ \delta_4 = ? \\ \delta_5 = ? \\ \delta_6 = ? \end{Bmatrix}_G$$

i.e.

$$\begin{Bmatrix} F_3 = 0 \\ F_4 = 0 \\ F_5 = P \\ F_6 = 0 \end{Bmatrix}_G = \begin{bmatrix} K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \delta_1 = 0 \\ \delta_2 = 0 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}_G$$

or bringing terms of boundary conditions to the force side :

$$\begin{Bmatrix} F_3^* = F_3 - k_{31}\delta_1 - F_{32}\delta_2 \\ F_4^* = F_4 - K_{41}\delta_1 - K_{42}\delta_2 \\ F_5^* = F_5 - K_{51}\delta_1 - K_{52}\delta_2 \\ F_6^* = F_6 - K_{61}\delta_1 - K_{62}\delta_2 \end{Bmatrix} = \begin{bmatrix} K_{33} & K_{34} & K_{35} & K_{36} \\ K_{43} & K_{44} & K_{45} & K_{46} \\ K_{53} & K_{54} & K_{55} & K_{56} \\ K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}$$

or $$\boxed{\underset{\text{Known}}{\{F^*\}} = [K^*]\underset{\text{unKnown}}{\{\delta\}}}$$

COMPUTER IMPLEMENTATION :

N_b = Total. no. of boundary conditions.
$k(j)$ = k-th d.o.f associated with j-th b.c.
$j = 1, 2, 3 \ldots N_b$

$\delta_k = a_{k(j)}$; Boundary value for kth d.o.f.

Modify force vector : $F_i^* = a_k$ if $i = k(j)$

if $i \neq k$ $\quad F_i^* = F_i - \sum_{j=1}^{N_b} \left(K_{ik(j)} \cdot a_{k(j)}\right)$

Modify stiffness matrix :

Initialize: $[K^*] = [K]$
Modify : $K_{kk}^* = 1.0$; $K_{ki}^* = K_{ik}^* = 0$ for $i \neq k$

Application to the cantilever:

$$\{F^*\}_G = \begin{Bmatrix} F_1 = a_1 \to 0 \\ F_2 = a_2 \to 0 \\ F_3^* = F_3 - K_{31}a_1 - K_{32}a_2 \\ F_4^* = F_4 - K_{41}a_1 - K_{42}a_2 \\ F_5^* = F_5 - K_{51}a_1 - K_{52}a_2 \\ F_6^* = F_6 - K_{61}a_1 - K_{62}a_2 \end{Bmatrix}$$

$$[K^*] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{33} & K_{34} & K_{35} & K_{36} \\ 0 & 0 & K_{43} & K_{44} & K_{45} & K_{46} \\ 0 & 0 & K_{53} & K_{54} & K_{55} & K_{56} \\ 0 & 0 & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix}$$

and solve for $\{\delta\}_G$ from

$$\{F^*\}_G = [K^*]\{\delta\}_G$$

(B) PENALTY METHOD.

Modification of $[K]_G$:

– A Large number C is added to the diagonal element, corresponding to the boundary d.o.f., viz. $k(j)$, $j = 1, 2, \ldots N_b$

$$\boxed{K^*_{kk} = K_{kk} + C}$$

– C should very large compared to K_{kk}

Modification of $\{F\}_G$:

Forces $\boxed{F^*_k = F_k + C \cdot a_k}$

With these modifications; the matrices for the two noded cantilever beam become :

$$\begin{bmatrix} (K_{11}+C) & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ & (K_{22}+C) & K_{23} & K_{24} & K_{25} & K_{26} \\ & & K_{33} & K_{34} & K_{35} & K_{36} \\ & \text{symmetric} & & K_{44} & K_{45} & K_{46} \\ & & & & K_{55} & K_{56} \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{Bmatrix} F_1 + Ca_1 \\ F_2 + Ca_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

Notice : Because of the dominance of C,

$$\sum_{j=1}^{6} K_{ij}\,\delta_j + C\delta_K = F_i + Ca_K$$

or

$$C\,\delta_K \cong Ca_K$$

$$\delta_K \approx a_K$$

V. SUPPORT REACTIONS:

- These correspond to those d.o.f.(s) that were prescribed 'a priori' before the solution process

- Along the k-th restrained d.o.f., the reaction force R_k is given as

R_k = Total nodal force along k d.o.f.

— Any applied ext. nodal force along k d.o.f.

$$\boxed{R_k = \sum_{j=1}^{N} K_{kj}\,\delta_j - F_k}$$

After solving for $\{\delta\}_G$, R_k values can be determined.

FIG 16

$w^{(j)}$: deflection from analysis using j elements

TABLE 1.1

DEFLECTION : (MM)

X=	$W_{ANALYTIC}$	$W^{(1)}$	$W^{(2)}$	$W^{(4)}$	$W^{(6)}$
0	0.	0.	0.	0.	0.
1.0	0.425				0.425
1.5	0.929			0.929	
2.0	1.601				1.601
3.0	3.378		3.378	3.378	3.378
4.0	5.605				5.605
4.5	6.841			6.841	
5.0	8.132				8.132
6.0	10.810	10.810	10.810	10.810	10.810

SLOPE : (RADIANS)

X	$\theta_{ANALYTIC}$	$\theta^{(1)}$	$\theta^{(2)}$	$\theta^{(4)}$	$\theta^{(6)}$
0	0.	0	0	0.	0.
1.0	0.825×10^{-3}				0.825×10^{-3}
1.5	1.182×10^{-3}			1.182×10^{-3}	
2.0	1.501×10^{-3}				1.501×10^{-3}
3.0	2.027×10^{-3}		2.027×10^{-3}	2.027×10^{-3}	2.027×10^{-3}
4.0	2.402×10^{-3}				2.402×10^{-3}
4.5	2.534×10^{-3}			2.534×10^{-3}	
5.0	2.627×10^{-3}				2.627×10^{-3}
6.0	2.703×10^{-3}	2.703×10^{-3}	2.703×10^{-3}	2.703×10^{-3}	2.703×10^{-3}

3.3 Equivalent nodal loads for the Euler beam element under distributed loading

How do we take care of distributed loading over the element?

We need to find equivalent nodal loads that replace the distributed loading.

For each element, the equivalent nodal loads for the distributed load are given by the set of forces, equal and opposite to the end support reactions on the same element in a fixed-fixed condition.

THE RATIONALE:

General: $[k^e]\{\delta^e\} = \{F^e\}$

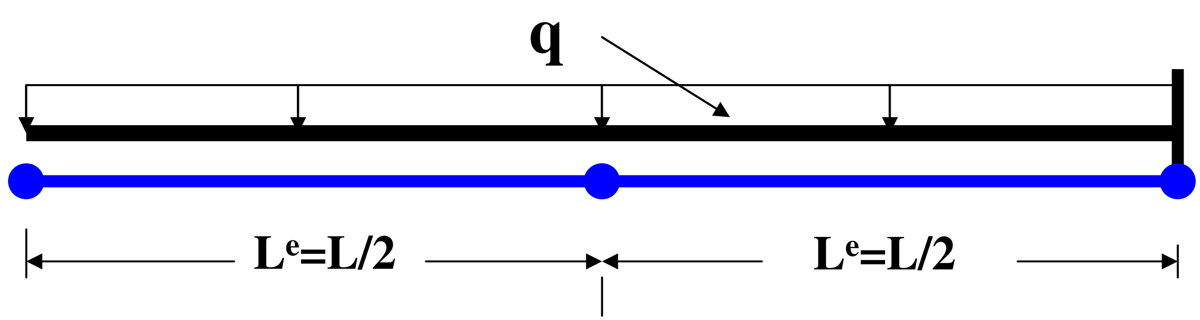
$$= \{F^e_{Applied}\} + \{R^e\}$$

where $\{R^e\}$ is the reaction force vector

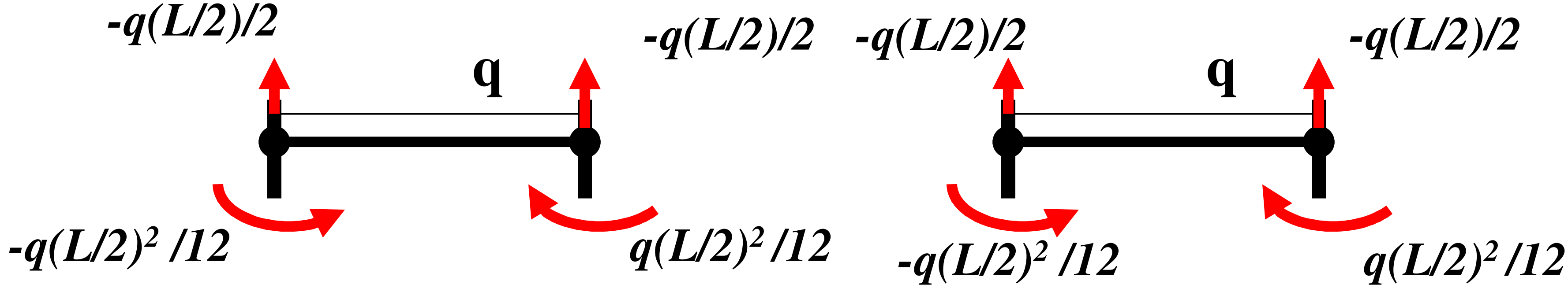
If the element is 'locked' at all nodal d.o.f.(s),

then $\{F^e\} = \{F^e_{Applied}\} + \{R^e\} = 0$

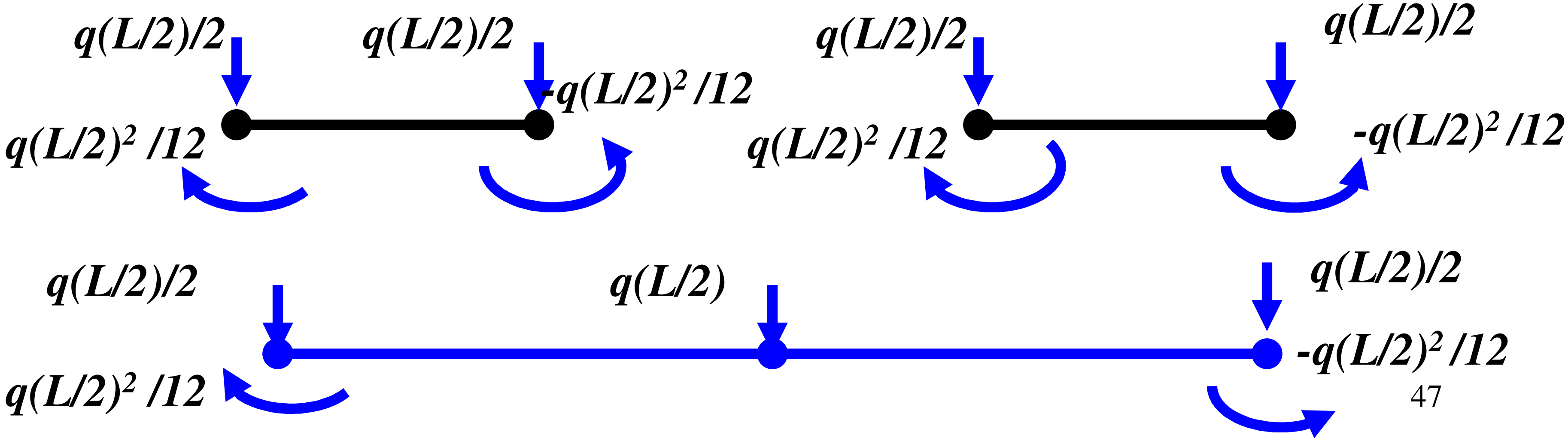
$$\Rightarrow \boxed{\{F^e\}_{eq.} = -\{R^e\}}$$



Fixed end reactions on element ends



Equivalent nodal loads



3.4 Strain and stress resultant in the linear Euler beam element

Element Strain

$$bending \quad strain \quad in \quad fibre(z), \qquad \varepsilon = \frac{z}{R}$$

$$bending \quad curvature \quad of \quad neutral-axis, \qquad \varepsilon^h = \frac{1}{R} = \frac{d^2 w^h}{dx^2}$$

$$\varepsilon^h = [B]\{\delta^e\} \qquad \textbf{(3.4)}$$

$$[B] = Strain - displacement \quad matrix$$

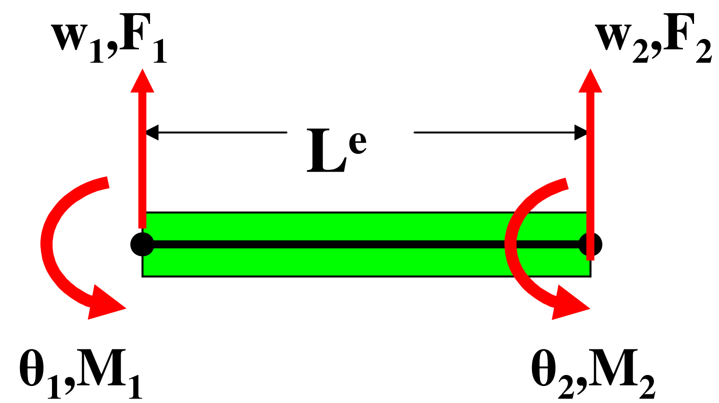
Element Stress Resultant

$$\{M^e\} = EI^e.[B]\{\delta^e\}$$

$$In \quad general \qquad \{\sigma^h\} = [D][B]\{\delta^e\}$$

$$\textbf{(3.5)}$$

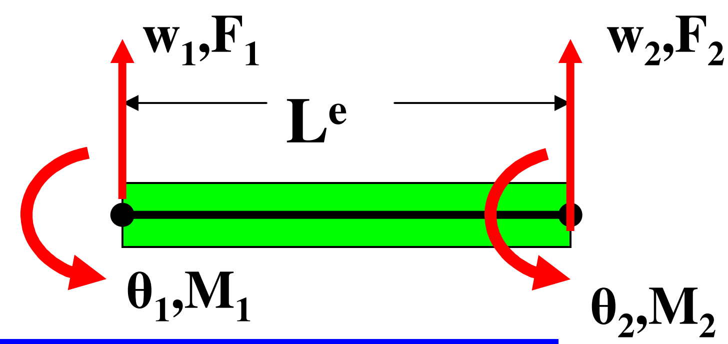
$$[D] = Element \quad elastic \quad rigidity \quad matrix$$



Governing differential equation for nodal (point) loads in FEA : → $$EI\frac{d^4w^h}{dx^4}=0 \quad (3.6)$$

$$w^h = ax^3+bx^2+cx+d$$

$$\varepsilon^h = \frac{1}{R} = \frac{d^2w^h}{dx^2}$$



Displacement (cubic polynomial of C^1 continuity):

$$w^h(\xi)=N_1(\xi)w_1+N_2(\xi)\theta_1+N_3(\xi)w_2+N_4(\xi)\theta_2 \qquad \xi=\frac{x}{L^e}$$

$$w^h(\xi)=[N_1(\xi)\quad N_2(\xi)\quad N_3(\xi)\quad N_4(\xi)]\{\delta^e\}=[N]\{\delta^e\} \quad (3.7)$$

Shape functions: $N_1(\xi)=1-3\xi^2+2\xi^3$ $\quad N_2(\xi)=L^e\xi(\xi-1)^2$

$N_3(\xi)=3\xi^2-2\xi^3$ $\quad N_4(\xi)=L^e\xi^2(\xi-1)$

Bending curvature (strain):

$$\varepsilon^h=\frac{d^2w^h(\xi)}{dx^2}=\frac{1}{L^{e2}}\frac{d^2w^h(\xi)}{d\xi^2}=\frac{1}{L^{e2}}\left[(-6+12\xi)\quad L^e(6\xi-4)\quad(6-12\xi)\quad L^e(6\xi-2)\right]\begin{Bmatrix}w_1\\ \theta_1\\ w_2\\ \theta_2\end{Bmatrix}$$

$$\boxed{\varepsilon^h=[B]\{\delta^e\}} \quad (3.8)$$

An example of a best fit function to a given function

Given Quadratic Curve is

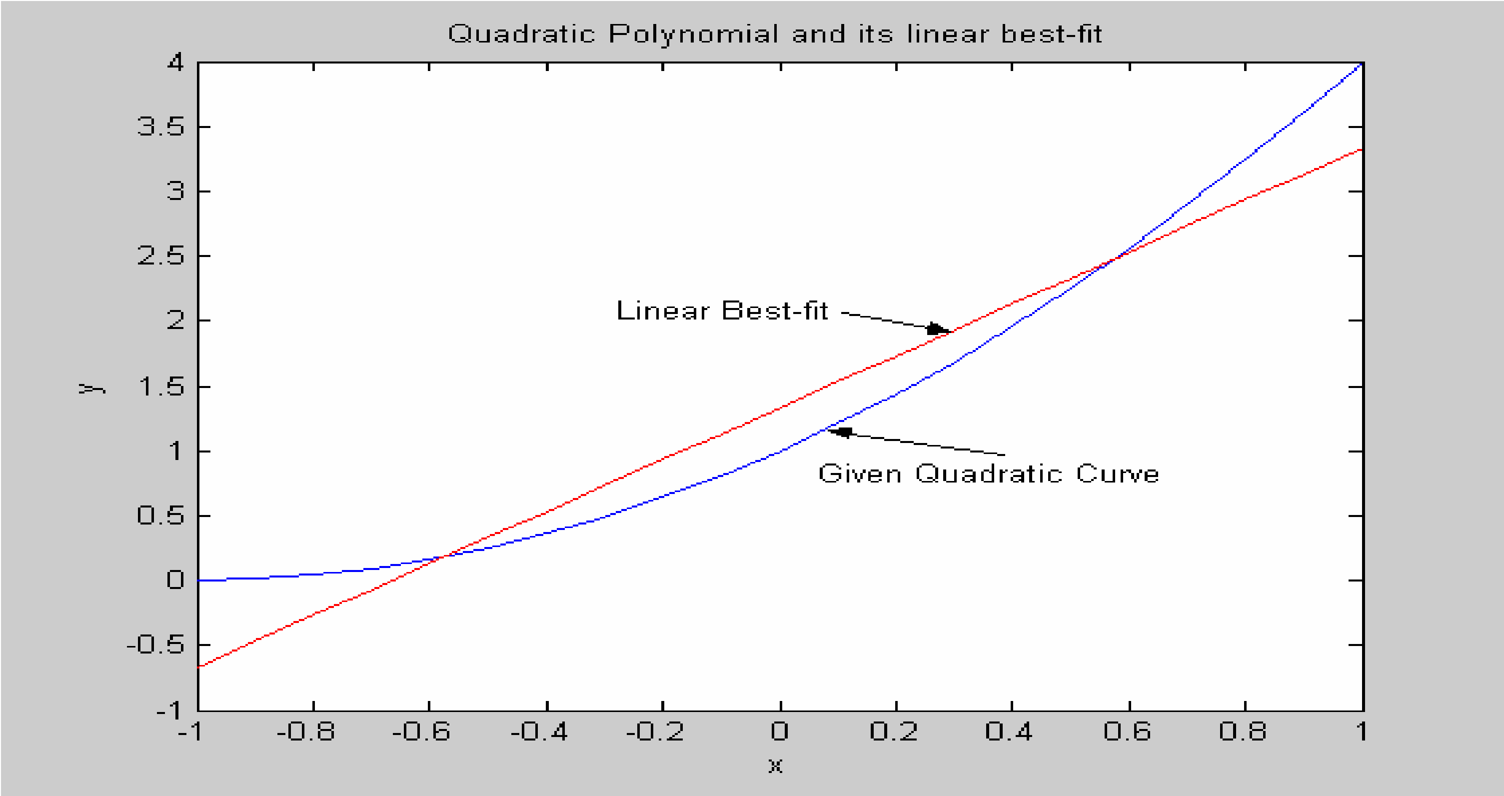
$$y = p_3 = 1 + 2\xi^* + \xi^{*2} = \frac{4}{3}P_1 + 2P_2 + \frac{1}{3}P_3 \qquad -1 \le \xi^* \le 1$$

$$P_1 = 1 \quad P_2 = \xi^* \quad P_3 = (3\xi^{*2} - 1) \quad \textit{are} \quad \textit{Legendre} \quad \textit{Polynomials}$$

$$\textit{Orthogonality}: \int_{-1}^{1} P_i P_j d\xi^* = 0 \quad \textit{for} \quad i \ne j$$

Linear Best-fit to y is

$$\bar{y} = \frac{4}{3} + 2\xi^* = \frac{4}{3} + 2P_2$$

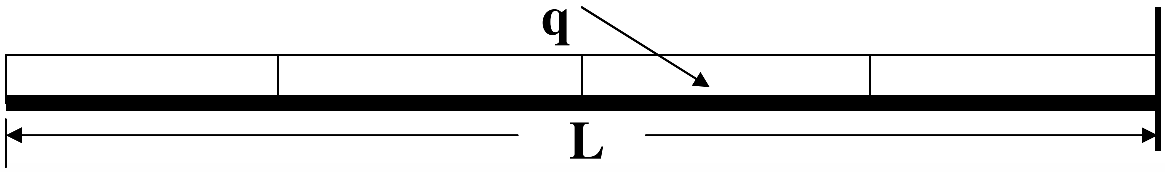


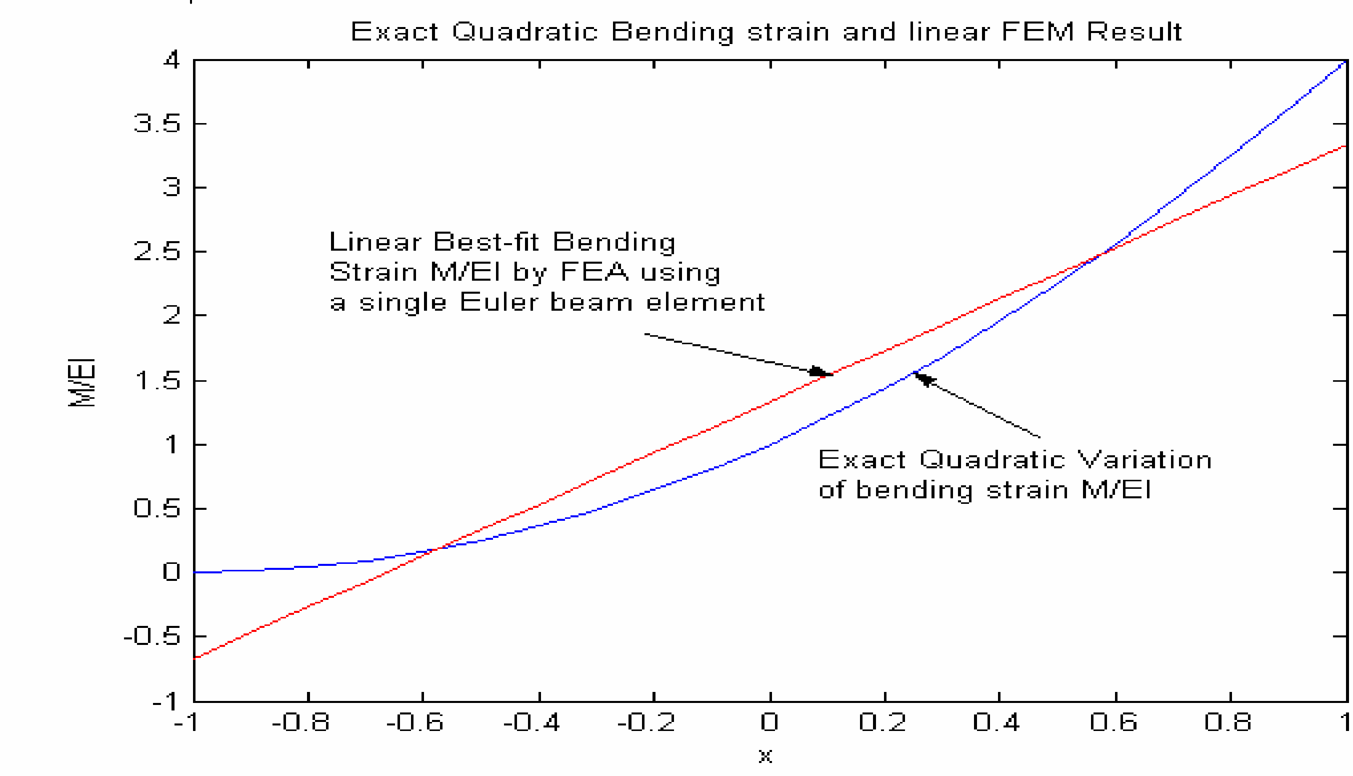
3.5 An indication of the best-fit rule in FEA …

CANTILEVER BEAM ANALYSIS USING A SINGLE EULER BEAM ELEMENT OF LENGTH L

Uniformly distributed loading is q per unit length.

q is such that fixed end curvature (bending strain) is $qL^2/2EI=4$ (m^{-1})

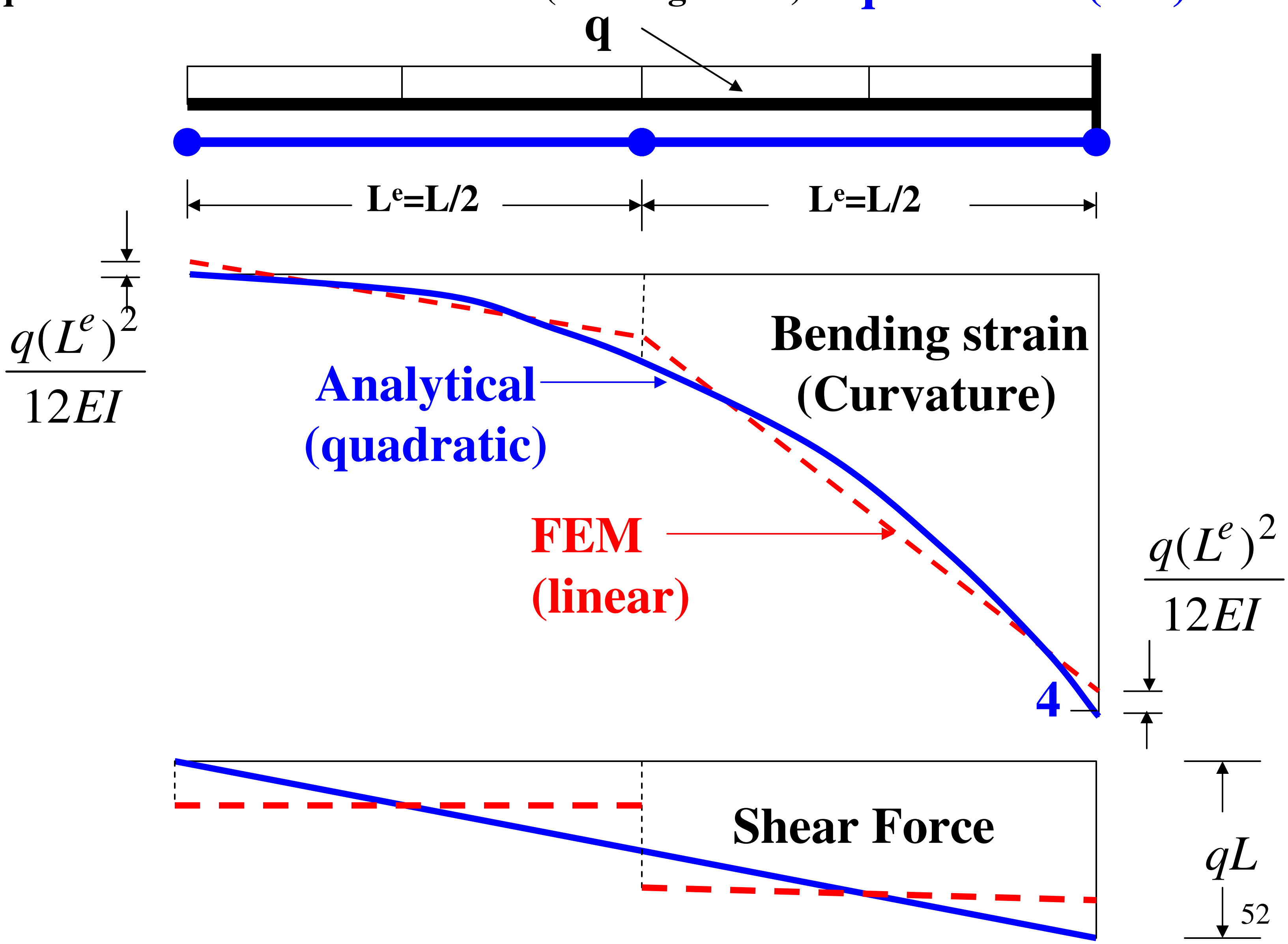




CANTILEVER BEAM ANALYSIS USING TWO EULER BEAM ELEMENTS

Uniformly distributed loading is q per unit length.

q is such that fixed end curvature (bending strain) is $qL^2/2EI=4\ (m^{-1})$



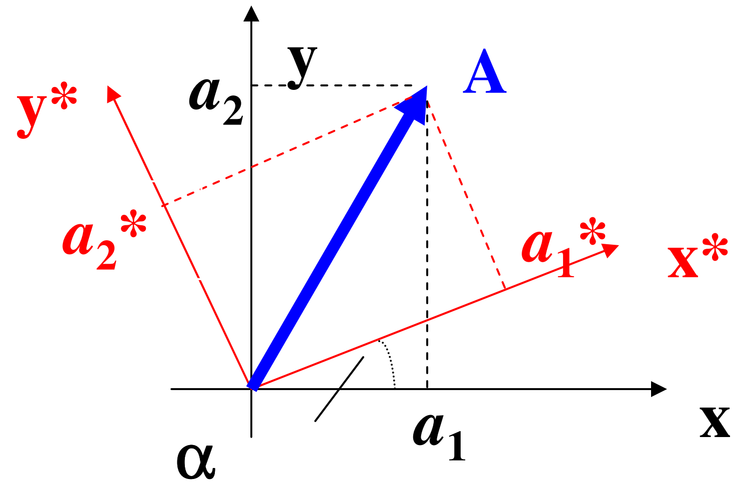
Lecture 1

Chapter 4

Transformation of co-ordinates

4.1 What is meant by transformation of co-ordinates ?

Consider a vector A in the plane, being observed from two reference frames S and S*



$$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} = a_1^{\,*}\mathbf{i}^* + a_2^{\,*}\mathbf{j}^*$$

$$\mathbf{A} = \begin{bmatrix}\mathbf{i} & \mathbf{j}\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\end{Bmatrix} = \begin{bmatrix}\mathbf{i}^* & \mathbf{j}^*\end{bmatrix}\begin{Bmatrix}a_1^{\,*}\\ a_2^{\,*}\end{Bmatrix}$$

$$\mathbf{A} = \begin{bmatrix}\mathbf{i} & \mathbf{j}\end{bmatrix}\{A\} = \begin{bmatrix}\mathbf{i}^* & \mathbf{j}^*\end{bmatrix}\{A^*\}$$

Using trigonometry:

$$\begin{Bmatrix}a_1^{\,*}\\ a_2^{\,*}\end{Bmatrix} = \begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\end{Bmatrix}$$

$$\boxed{\{A^*\} = [T]\{A\}} \qquad \textbf{(4.1)}$$

Invariance of magnitude of a vector upon transformation

$$\{A^*\}^T\{A^*\} = \{A\}^T\{A\}$$

$$\{A\}^T[T]^T[T]\{A\} = \{A\}^T\{A\}$$

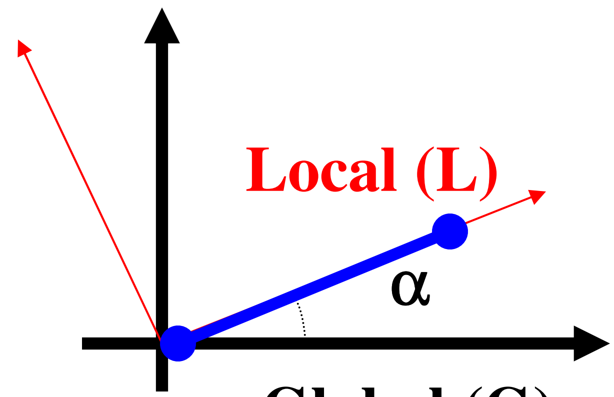
$$\Rightarrow [T]^T[T] = [I]$$

The transformation matrix is orthogonal $[T]^T = [T]^{-1}$ **(4.2)**

4.2 Transformation of element matrices

It is often required to transform the equilibrium equations of an element from its own local co-ordinates to a global co-ordinate system, common to all elements of the domain.

$$\boxed{\{F^e{}_L\} = [K^e{}_L]\{\delta^e{}_L\}}$$ **in local co-ordinates**



$[T^e]$ is the element transformation matrix

$$\{F^e{}_L\} = [T^e]\{F^e{}_G\} \qquad \{\delta^e{}_L\} = [T^e]\{\delta^e{}_G\}$$

$$[T^e]\{F^e{}_G\} = [K^e{}_L][T^e]\{\delta^e{}_G\}$$

Pre-multiply both sides by $[T^e]^T$

$$[T^e]^T[T^e]\{F^e{}_G\} = [T^e]^T[K^e{}_L][T^e]\{\delta^e{}_G\}$$

$$\{F^e{}_G\} = [T^e]^T[K^e{}_L][T^e]\{\delta^e{}_G\}$$

because $[T^e]^T = [T^e]^{-1}$

Transformation rule

$$[K^e{}_G] = [T^e]^T[K^e{}_L][T^e]$$

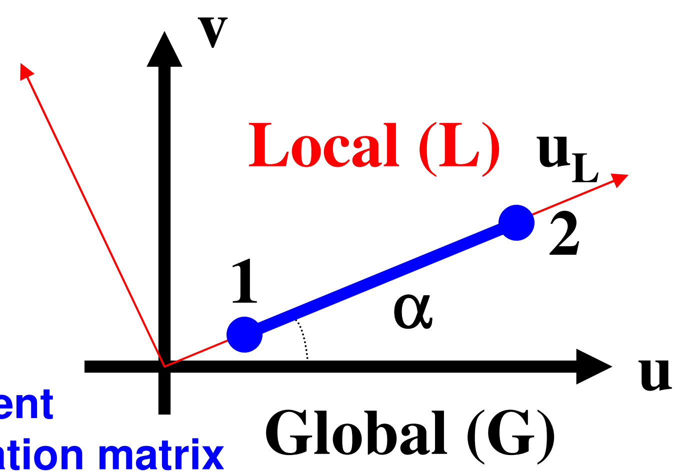
$$\{F^e{}_G\} = [T^e]^{-1}\{F^e{}_L\} = [T^e]^T\{F^e{}_L\}$$

$$\boxed{\{F^e{}_G\} = [K^e{}_G]]\{\delta^e{}_G\}}$$ **in global co-ordinates** **(4.3)**

4.3 Transformation of the bar (truss) element (in the plane)

$$\boxed{\{F^e{}_L\} = [K^e{}_L]\{\delta^e{}_L\}}$$

in local co-ordinates



$$\begin{Bmatrix} F^e{}_{1,L} \\ 0 \\ F^e{}_{2,L} \\ 0 \end{Bmatrix} = \frac{EA^e}{L^e}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u^e{}_{1,L} \\ v^e{}_{1,L} \\ u^e{}_{2,L} \\ v^e{}_{1,L} \end{Bmatrix}$$

[T^e] : element transformation matrix

Transformation rule

$$[K^e{}_G] = [T^e]^T[K^e{}_L][T^e]$$

$$\{F^e{}_G\} = [T^e]^{-1}\{F^e{}_L\} = [T^e]^T\{F^e{}_L\}$$

$$[T^e] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$\boxed{\{F^e{}_G\} = [K^e{}_G]]\{\delta^e{}_G\}}$$

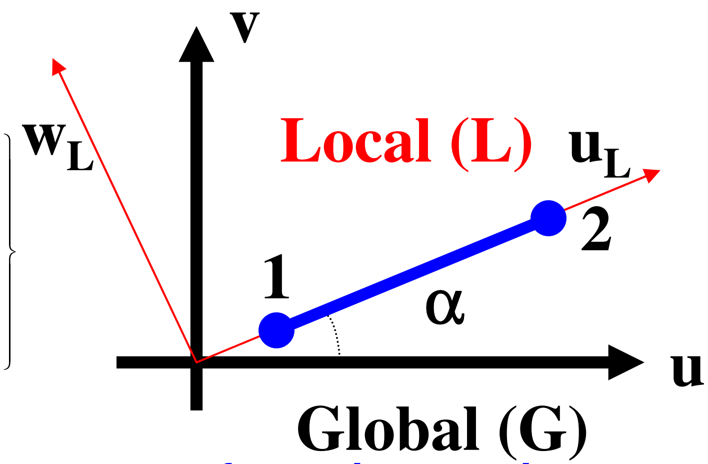
in global co-ordinates

$$[T^e]^T = [T^e]^{-1}$$

$$\begin{Bmatrix} u^e{}_{1,L} \\ v^e{}_{1,L} \\ u^e{}_{2,L} \\ v^e{}_{2,L} \end{Bmatrix} = [T^e]\begin{Bmatrix} u^e{}_{1,G} \\ v^e{}_{1,G} \\ u^e{}_{2,G} \\ v^e{}_{2,G} \end{Bmatrix} \qquad \textbf{(4.4)}$$

4.4 Transformation of the frame element (in the plane)

$$\boxed{\{F^e{}_L\} = [K^e{}_L]\{\delta^e{}_L\}}$$ **in local co-ordinates**



$$\begin{Bmatrix} F_{x1,L} \\ F_{y1,L} \\ M_{1,L} \\ F_{x2,L} \\ F_{y2,L} \\ M_{2,L} \end{Bmatrix} = \begin{bmatrix} EA^e/L^e & 0 & 0 & -EA^e/L^e & 0 & 0 \\ 0 & 12(EI^e/L^{e^3}) & 6(EI^e/L^{e^2}) & 0 & -12(EI^e/L^{e^3}) & 6(EI^e/L^{e^2}) \\ 0 & 6(EI^e/L^{e^2}) & 4(EI^e/L^e) & 0 & -6(EI^e/L^{e^2}) & 2(EI^e/L^e) \\ -EA^e/L^e & 0 & 0 & EA^e/L^e & 0 & 0 \\ 0 & -12(EI^e/L^{e^3}) & -6(EI^e/L^{e^2}) & 0 & 12(EI^e/L^{e^3}) & -6(EI^e/L^{e^2}) \\ 0 & 6(EI^e/L^{e^2}) & 2(EI^e/L^e) & 0 & -6(EI^e/L^{e^2}) & 4(EI^e/L^e) \end{bmatrix} \begin{Bmatrix} u_{1,L} \\ w_{1,L} \\ \theta_{1,L} \\ u_{2,L} \\ w_{2,L} \\ \theta_{2,L} \end{Bmatrix}$$

Transformation rule

$$[K^e{}_G] = [T^e]^T [K^e{}_L][T^e]$$

$$\{F^e{}_G\} = [T^e]^{-1}\{F^e{}_L\} = [T^e]^T\{F^e{}_L\}$$

$[T^e]$: element transformation matrix

$$[T^e] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\{F^e{}_G\} = [K^e{}_G]]\{\delta^e{}_G\}}$$

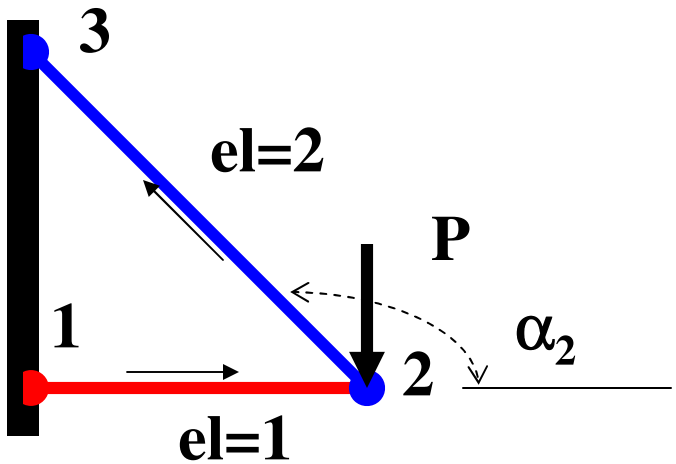
in global co-ordinates

$$[T^e]^T = [T^e]^{-1}$$

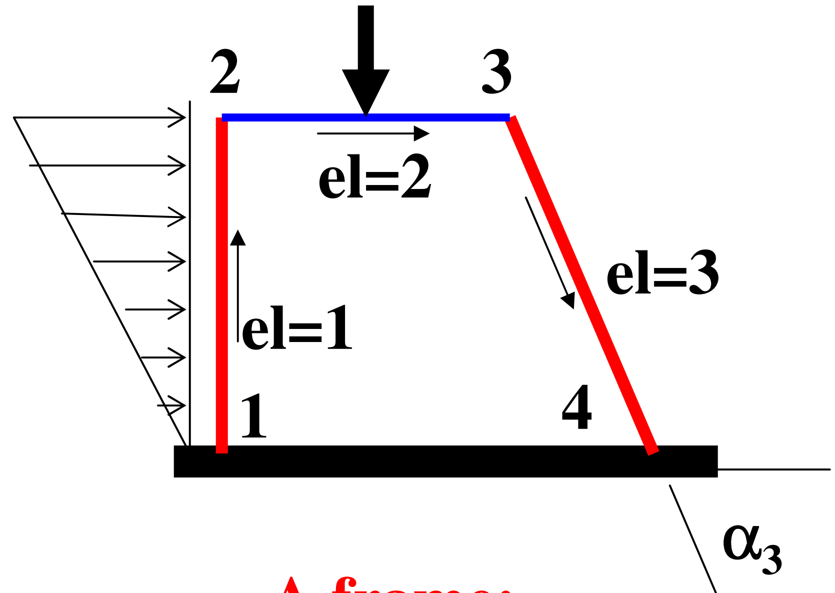
$$\begin{Bmatrix} u_{1,L} \\ w_{1,L} \\ \theta_{1,L} \\ u_{2,L} \\ w_{2,L} \\ \theta_{2,L} \end{Bmatrix} = [T^e] \begin{Bmatrix} u_{1,G} \\ v_{1,G} \\ \theta_{1,G} \\ u_{2,G} \\ v_{2,G} \\ \theta_{2,G} \end{Bmatrix} \qquad \textbf{(4.5)}$$

4.5 Analysis of trusses and frames

Transformation matrices are obtained from element orientations



A truss:
Assembly of axially loaded members.



A frame:
Assembly of bending and axially loaded members.